Inverse Lévy subordination in option pricing

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Overview

1. Time changes using inverse Lévy subordinators
2. Analytical theory. Fractional diffusions and CTRWs limits
3. Transform analysis and pricing
4. Time changes and measure change
5. Financial applications
   - Trading suspensions
   - Time multiscaling and illiquidity/liquidity transitions
   - Investor inertia and "flat volatiliy" modelling

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1. Time changes using inverse Lévy subordinators

Take a semimartingale return model X_t on a filtered space and an increasing process T_t.

The time changed process X_{T_t} is popular in financial modelling of returns/prices.

1. If X_t and T_t are Lévy and independent $\rightarrow$ X_{T_t} is Lévy . If X_t is a Brownian Motion business time interpretation (Carr, Madan, Yor, Geman, Barndorff-Nielsen etc..)
2. If T_t is absolutely continuous and increasing: $\rightarrow$ stochastic volatility. Dubins-Schwarz theorem or stochastic Lévy volatility

Critical properties above: either T_t-continuity of X_t or independence.

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Can one go beyond these two ideas? And if so, what would be the point for financial modelling?

1. Time changes using inverse Lévy subordinators

Suppose we have an increasing Lévy process (subordinator) L_t. Define:

$$H_t = inf\{s : L_s > t\}$$

the first hitting time of t or the inverse process to H_t.

- H_t is increasing. If L_t is strictly increasing (so L_t no driftless CPP) then H_t is continuous and any process X_t is H_t-continuous.
- Thus X_{H_t} inherits semimartingale properties from H_t (Jacod 1979) $\rightarrow$ change to EMM possible (in principle..) by fiddling with the chasracteristics.
- X_{H_t} is not Markovian and has non stationary increments;
- In some cases X_{H_t} can show long range dependence
- H_t and X_t do not need to be independent!

Good. But what for?

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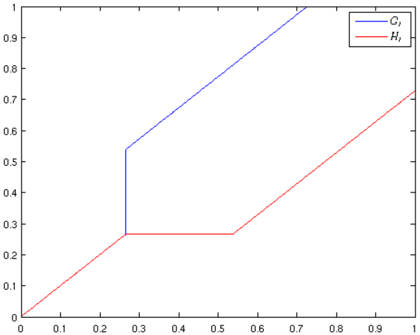

H_t when L_t is a CPP with drift

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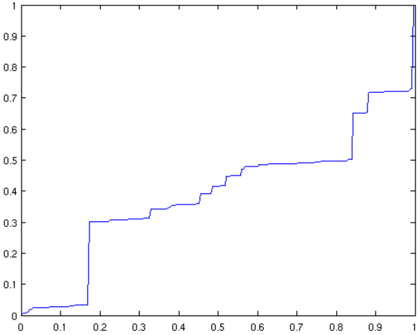

H_t when L_t is a driftless α-stable subordinator. In this case H_t is neither Lévy nor Lebesgue- absolutely continuous: purely singular time change! (compare with 1. and 2. from first slide).

Processes of the form X_{H_t} are actually very well reasearched.

Technicalitites

- The Caputo derivative ∂_t^α of order $\alpha < 1$ of a positive function $f(t)$ is:

$$\partial_t^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty f'(u)(t-u)^{-\alpha} du$$

- If $\hat{\cdot}$ is the Fourier transform observe by convolution:

$$\partial_t^\alpha \hat{f}(s) = s^\alpha \hat{f}(s) - s^{\alpha-1} f(0+)$$

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2. Analytical theory

Example 1 $X_t = t$, L_t standard α-stable subordinator (char func: $\exp(-s^{\alpha})$). H_t is an inverse α-stable subordinator.

- We known
$$\mathbb{E}[e^{-sT_t}] = E_\alpha(-st^\alpha)$$
with
$$E_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(1+\alpha j)}$$
the Mittag-Leffler function.
- Laplace transforming in t: $E_\alpha(\hat{-}st^\alpha) = y^{\alpha-1}/(y^\alpha + s)$ so that:
$$y^\alpha E_\alpha(\hat{-}st^\alpha) - y^{\alpha-1} = -sE_\alpha(\hat{-}st^\alpha)$$
- inverting the double Laplace transform and using the relation in the slide before we see that, the density $p(t,x)$ solves:
$$\partial_t^\alpha p(t,x) = \partial_x p(t,x)$$

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$$\partial_t^\alpha p(t,x) = \partial_x p(t,x)$$

Example 2 X_t is a Poisson process, L_t an independent α-stable subordinator. X_{H_t} is the fractional Poisson process.

- Equivalently: a renewal process with Mittag Leffler distributed waiting times of parameter λ (Poisson $\to$ exponential-λ waiting).
- Its density $p(t,x)$ solves the master PIDE.

$$\partial_t^\alpha p(t,x) = \lambda \int_{\mathbb{R}} (p(t,x+y) - p(t,x)) f(y) dy$$

where f is the jump distribution.

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2. Analytical theory

Example 3 More fractional diffusions equations arise as stable scaling Limit of CTRWs.

- Let T_i i.i.d. waiting times, Y_i returns (possibly coupled), set $N_t = max\{n : \sum T_i < t\}$ and consider the CTRW S_{N_t}.
- If $Var[Y_i] < \infty$, $\mathbb{E}[T_i] < \infty$ $T_i \sim \exp(\lambda)$ then Renewal theorem+ CLT:

$$c^{-1/2} S_{[cN_t]} \to \lambda W_t$$

.

- if $\mathbb{P}(Y_i > r) \sim r^{-\alpha}$, $\mathbb{P}(Y_i > r) \sim r^{-\beta}$, $0 < \alpha < 2$, $0 < \beta < 1$ then (Meerscahert Scheffler 2004)

$$c^{-\beta/\alpha} S_{[cN_t]} \to X_{H_t}$$

with X_t an α-stable process and L_t a β-stable subordinator. If Y_i and T_i are dependent, so are X_t and L_t.

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2. Analytical theory

Interpretation of S_{N_t}: tick-by-tick model of trade arriving at power-law distributed times and with possbily infinite variance returns. Distributions can be coupled. Very realistic!

- In the independent case the densitxy $p(x,t)$ of X_{H_t} satisfies (again Meerschaert Scheffler 2004)

$$\partial_t^\beta p(t,x) = \lambda \partial_{|x|}^\alpha p(t,x)$$

  for $\lambda > 0$ depends on T_i (e.g. M-L parameter), some operator $\partial_{|x|}^\alpha$ characterized by $\partial_{|x|}^\alpha \hat{p(t,x)} = |s|^\alpha \hat{p}(t,x) - \delta_x t^{-\beta}/\Gamma(1-\beta)$ and coinciding with the Caputo derivatives if $p > 0$. Sometimes $p(t,x)$ is known explicitly.
- $X_{H_t} = t^{\beta/\alpha} X_{H_1}$ i.e. selfsimilar with Hurst exponent β/α. Non gaussian, nonstationary, selfsimilar process ($\to$ compare against: fractional Brownian Motion).

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Theorem (Meerschaert and Scheffler 2009)

Let X_t, L_t be independent with Fourier and Laplace exponents ψ_X ϕ_L. Then if $\mathcal{L}_X$ and $\mathcal{L}_L$ be the generator of the convolution semigroups T_t^L and T_t^X associated with X_t andL_t. Then the density $p(t,x)$ of X_{H_t} satisfies in the mild sense the abstract Cauchy problem:

$$C(\mathcal{L}_L)p(x,t) = \mathcal{L}_X p(x,t)$$

where $C(\mathcal{L}_L)$ is the Caputo generalized operator defined by

$$C(\hat{\mathcal{L}}_L)f(x,s) = \hat{\mathcal{L}_L}f(x,s) - s^{-1}\phi_L(s)\hat{f}(x,0) \tag{1}$$

$$= \phi_L(s)\hat{f}(x,s) - s^{-1}\phi_L(s)\hat{f}(x,0) \tag{2}$$

A general version for dependent X_t, L_t exists but is not as nice.

3. Transform analysis and pricing

This analytic theory makes also possible the characteristic function analysis. (again Meerschaert and Scheffler 2009)

Let $\psi_{X,L}(z,t)$ be the Fourier-Laplace epxonent of X_t, L_t:

$$\int_0^\infty e^{-st}\mathbb{E}[e^{izX_{T_{H_t-}}}]dt = \frac{1}{s}\frac{\phi_L(s)}{\psi_{X,L}(z,s)}$$

when either :

1. X_t are independent
2. L_t has infinite activity and no drift.

Important! in case 2 by stochastic continuity $X_{H_t} = X_{H_t-}$

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This analytic theory makes also possible the characteristic function analysis. (again Meerschaert and Scheffler 2009)

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2. In the dependent case, proof valid only if L_t is driftless. Whether this is a technical or structural is an interesting question. For example when $X_t = L_t$ this assumpiton can be relaxed.

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4. Time changes and measure changes

Let X_t be a process adapted to a time change T_t and assume X_t and T_t to be independent.

- Z_t is a martingale density for X_t. Then Z_{T_t} is a martingale density for X_{T_t} inducing $\mathbb{Q}^Z \sim \mathbb{P}$. (Jacod, Jacod and Shiryaev)
- However we can also do a change measure on H_t!. If Y_t is a martingale density for T_t inducing $\mathbb{Q}^Y \sim \mathbb{P}$ then:

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1. Scalas, Gorenflo and Mainardi (2001): modeling of tick-by-tick intra-day trades using the fractional equation corresponding to X_t α Lévy, stable, and L_t, β-stable subordinator. "Phenomenological" model. No option pricing, no martingale relations.
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5A. Time multiscaling

Assume X_t Lévy and L_t (indep.) be tempered standard stable subordinator: parameters $\alpha < 1$, $\lambda > 0$. Lévy measure

$$\nu(dx) \sim C\frac{e^{-\lambda x}}{x^{1+\alpha}} I_{\{x>0\}} dx$$

- α Selfsimilarity parameter. Inverse of Hurst exponent, "power law" behaviour.
- λ "cutoff parameter". Decreases the rate of occurrence of the big jumps and makes moments finite

Thus $log S_t = X_{H_t}$ can be used to model an asset with various degrees of delay of the trading activity at different time scales i.e. "time multiscaling":

- Fixed t: higher λ, less incidence of long waiting times, more akin to Lévy activity.
- Fixed λ: longer time scale, increased resemblance to a diffusion $\rightarrow$ by finite variance/CLT.

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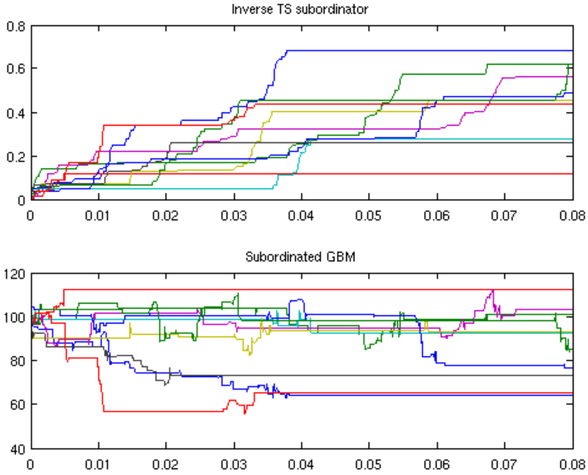

$T = 0.1$, $\lambda = 1$, $\alpha = 0.5$. Short term stale tade arrivals

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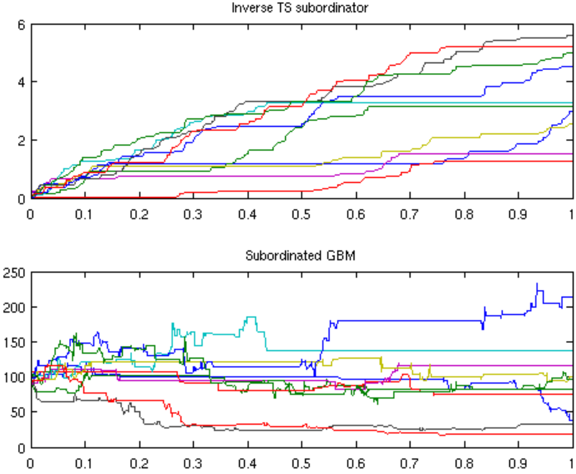

$T = 1$, $\lambda = 1$, $\alpha = 0.5$. The situation does not improve much at longer horizon.

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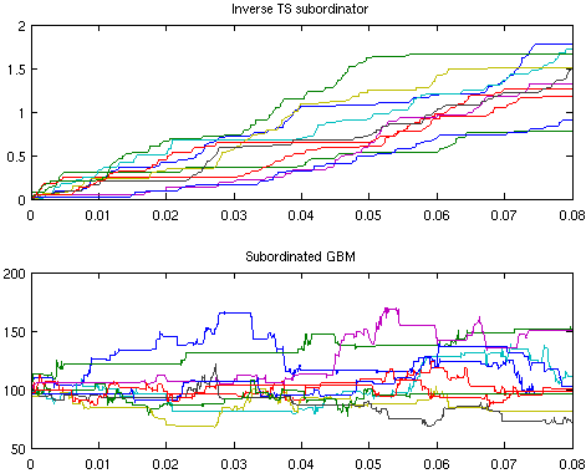

$T = 0.1$, $\lambda = 50$, $\alpha = 0.5$. Not that different from the case $\lambda = 1$

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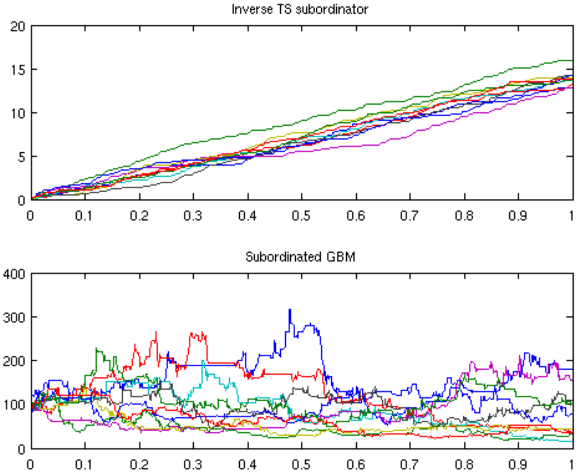

$T = 1$, $\lambda = 50$, $\alpha = 0.5$. Now trade is much more fluid.

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- The process is non-Markovian and has a marked long-rang dependence (Leonenko et. al)
- When $\lambda = 0$ the Laplace inversion yielding to the model c.f. can be performed exactly, and involves the Mittag Leffler function
- The parameter λ when calibrated to option prices (remember!) gives the price of market latency risk, which is the compensation investor should require to face the risk an illiquid asset does not become liquid as time goes on.
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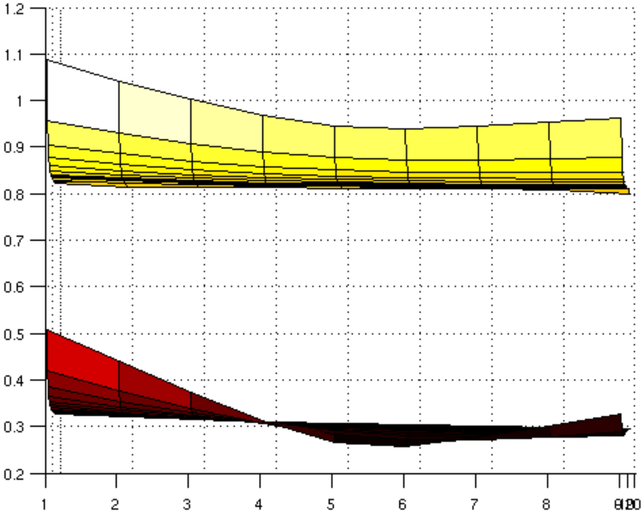

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- We can model transitions from illiquidity to liquidity at a fixed time scale, by introducing an exponential time ξ, of parameter β i.e.

$$\log S_t = X_{H_t} I_{\{t<\xi\}} + X_t I_{\{t>\xi\}}$$

which is analytically tractable because

$$\mathbb{E}[e^{iz\log(S_t)}] = \mathbb{E}[e^{iz\log(X_{H_t})}]e^{-\beta} + e^{t\psi_X(z)}(1-e^{-\beta})$$

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5B. Trading suspensions

Problem: to a stock with suspection one cannot apply directly the no-arbitrage principle because it is not traded at all times

Idea: separate fundamental value and market quote and require by market efficency that they coincide when the market is open.

1. X_t is the trade noise and R_t the non-trade noise ("market gossip")
2. L_t is a rate λ CPP with drift 1 and $exp \sim \beta$ jump sizes (linear time+exponential suspension waiting)
3. Define the fundamental value $\log S_t = X_{H_t} + R_t + rt$
4. Introduce a second time change: $\tau_t = L_{H_t-}$ is the last instant of quote update before t.
5. Define the quote process as the time change $Q_t = S_{\tau_t}$

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- Rather, the derivative should reference in case at matruity the asset is suspended, $Q_t +$ interests: i.e. pay on the "forward" F_t

$$F_t = e^{r(t-\tau_t)} Q_t$$

- Note that $e^{-rt} F_t$ being a martingale is equivalent to $e^{-r\tau_t} Q_t$ being a martingale $\rightarrow$ "stochastic discounting principle for assets with suspensions"
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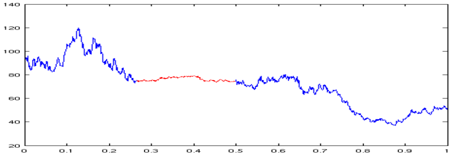
- Rather, the derivative should reference in case at matruity the asset is suspended, Q_t+ interests: i.e. pay on the "forward" F_t

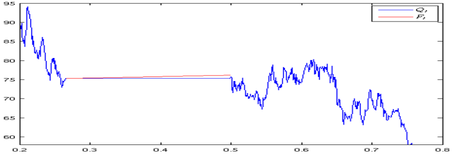
$$F_t = e^{r(t-\tau_t)} Q_t$$

- Note that $e^{-rt} F_t$ being a martingale is equivalent to $e^{-r\tau_t} Q_t$ being a martingale $\rightarrow$ "stochastic discounting principle for assets with suspensions"
- λ and β are the parameters in H_t for the market price of suspension risk
- Volatility surface: λ, and β steepen further the short term skew $\Rightarrow$ impact of trading halts on weekly option pricing!

5B. Trading suspensions

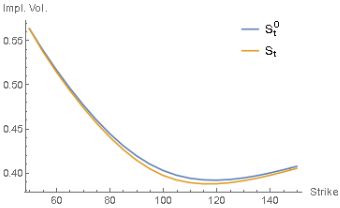


S_t. In red the suspension, fundamental value evolves only due to R_t

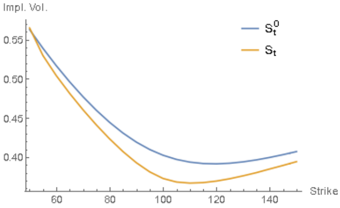


Q_t and F_t. The discounted value $e^{-rt}Q_t$ is not a martingale, but $e^{-rt}F_t$ is.

5B. Trading suspensions

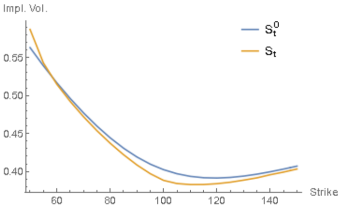


X_t CGMY, compared to F_t. $\lambda = 2$, $\beta = 50$, monthly. Excess skew observed.

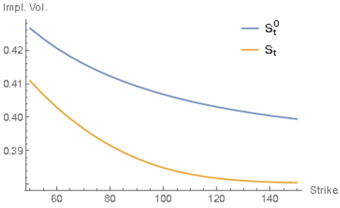


$\lambda = 12$, $\beta = 50$, monthly. Increasing jump frequency, further steepening

5B. Trading suspensions



$\lambda = 2$, $\beta = 12$, monthly. Increasing jump duration, same effect



$\lambda = 2$, $\beta = 12$, yearly. Which seems to persist at longer maturities.

5C. Flat volatility, investors inertia

Consistently with behavioural models, volatility series of some asset show periods of staleness punctuated by bursts of activity (Ghoulmie et al. 2005)

$\Rightarrow$ Idea: use H_t to flatten volatiliy

Take X_t, v_t driven by an SDE involving correlated B.M W_t^1, W_t^2 and an inverse subordinator to L_t to produce the time changed SDE for v_{H_t}.

- Time-changed stochastic calculus available (Kobayashi 2010);
- Feynman Kac theorems and series expressions for the density function $p(t, x)$ of v_{H_t} available (Leonenko, Meerschaert, Sirjoski 2013)

This model has long memory. Furthermore I expect it to be able to reproduce short ATM skew (sheer intuition, and comparison with the other two solutions) and maybe this can be quantitatively assessed using martingale expansions (Aloset al. 2006, Fukusawa 2010) $\Rightarrow$ competitor of rough volatiltiy?

Problem: analytical tractability?

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Thanks for the attention

(input, crticism and cooperation appreciated!)

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