Target Volatility Option Pricing *

Lorenzo Torricelli †

Department of Mathematics

Imperial College London

London SW7 2BZ, United Kingdom

September 18, 2009

Abstract

In this paper we derive several pricing methods for a new kind of *volatility-based* European-style option, the target volatility option (TVO). A TVO pays at maturity a proportion of a vanilla European call option based on the ratio between a specified target parameter $\overline{\sigma}$ and the average realised volatility of the underlying. Three main techniques are proposed: a local at-the-money power series expansion; a Laplace and Fourier transform method; approximations via uniform and L^2 convergence of replicating claims. Our results hold true in a stochastic volatility model under the usual independence conditions. Numerical evidence supporting our results is provided.

*Report submitted in partial fulfilment of the degree of MSc in Mathematics and Finance at Imperial College London. CID 00575562.

†mr_3_7@hotmail.it

A mio padre

Contents

1 Preliminaries and assumptions

1.1 Introduction

Quadratic variation and volatility based derivatives trading first took off in the late nineties following an unprecedented increase of implied volatility levels. Since then volatility products become a very liquid and widely traded instrument; from the market participants' perspective volatility derivatives are to be used either to hedge away volatility risk or to speculate on future realised volatility. As a result a wide literature on the subject arose, (Carr and Madan [6], Fritz and Gatheral, [9], the book by Gatheral [10] and Carr and Lee [4], just to list a few) and volatility derivatives became a very well understood product relying on a solid theoretical basis.

A more recent innovation, dating back to the past few years, has been the introduction of derivative products paying on an underlying traded asset *and* the realised volatility of the asset itself. In other words the contingent claims associated with these derivatives are *joint functions* of the asset value and the realised volatility of the asset throughout the life of the contract. To our knowledge, a rigorous theory for pricing and hedging these instruments has yet to be developed. Indeed, valuation of these products in the classical setting is, in its full generality, a highly non-trivial problem. Risk-neutral expectations of such claims depend on the joint distribution of two heavily mutually dependant variables, as the asset and its quadratic variation are, even in the case in which simple uncorrelated dynamics are assumed for the relevant processes.

Target volatility options, (TVO in short) introduced in 2008, represent one such a kind of derivative. This contract is a volatility-based variation of a plain vanilla European option. A target volatility call option is written on an underlying asset S_t and gives the buyer the right to buy a certain fractional amount c of shares of S_t at a future date T for a price cK, where K is some fixed strike price. The amount c of shares to be exchanged for cK is the ratio of a specified constant $\overline{\sigma}$, and the average realised volatility of S_t during the lifespan of the contract, and it is a random variable.

The rationale behind the creation of target volatility options is better explained with an example. Assume an investor believes that the market implied volatility for a given option does not reflect his/her expectation of the future realised volatility of the underlying. In particular, assume that the investor is of the view that the future realised volatility of the stock will be lower than implied by option prices. Then he may enter a target volatility contract and choose the threshold $\bar{\sigma}$ to be his/her prediction of the future (average) realised volatility of the underlying. The price paid for the target volatility option is an (increasing) function of $\bar{\sigma}$ and will be typically lower than the corresponding vanilla contract. However, if the investor's prediction is correct the pay-off of the two options will be the same (see Section 2 for details).

In this paper we begin to work on the problem of pricing this option in the risk-neutral valuation setting. For simplicity we assume that no interest rates are paid in the market; furthermore we assume *independence* between the quadratic variation and the underlying. Even though this is a quite unrealistic assumption it is necessary to begin with it in our study; under such an assumption we will be able to reduce the expected value of the option as the expected value of a claim on the quadratic variation *alone* (Proposition 1.2). Having done so, the cornerstone on claims on volatility laid by Carr and Lee ([4]) will lead us on our way to pricing via replication through contingent claims on S_t.

After introducing the definitions and the setting in which we are going to work, we tackle the problem in three different ways. In Section 2 we begin by observing that an at-the-money TVO has nearly the same price as an European call option of constant volatility $\overline{\sigma}$, and then we deduce its value in a neighbourhood of S_0 by expanding the payoff in its Taylor series in K. In Section 3 we derive exact formulas for the Laplace and Fourier transforms of claims approximating the option and then find the value via inversion. In Section 4 we express the value as a limit of expectations of uniform and L^2 sequences converging to the claim. In Section 5 we give some numerical examples.

I would like to thank Giuseppe Di Graziano at Deutsche Bank AG Commodities Structuring for providing me with all the necessary background and helping me throughout this project.

1.2 The framework

Let us start by introducing the setting and definitions of our framework. The lifespan of contracts runs in a bounded closed interval $[0, T]$, $T < +\infty$ representing the maturity date. Pricing is always referred to time 0 (initiation of the contract), even though by considering conditional expectations most our results could be extended for time $t > 0$ evaluation. We assume throughout that there exist riskless market securities (bonds) on which no interest rate is paid.

Our market is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions. We assume that there exists a $\mathbb{P}$-equivalent measure $\mathbb{Q}$ under which any non-dividend-paying stock process S_t satisfies a stochastic equation of the form

$$dS_t = \sigma_t S_t dW_t, \quad t \in (0,T], \quad S_0 \in \mathbb{R}^+ \tag{1.1}$$

for a $\mathbb{Q}$-Brownian motion W_t and a stochastic process $\sigma_t > 0$ independent of W_t.

Unless specified otherwise we assume σ_t to be almost surely bounded in $[0,T] \times \Omega$, that is, there exists $m > 0$ such that

$$\sigma_t(\omega) < m, \tag{\textbf{A}}$$

for all $t \in [0,T]$ and all $\omega \in \Omega$ outside a $\mathbb{Q}$-null set.

σ_t is the *volatility* of S_t. Typically σ_t satisfies certain diffusion equations of the type

$$d\sigma_t = \mu_t(\sigma_t, t)dt + \nu_t(\sigma_t, t)dZ_t, \quad \sigma_0 > 0 \tag{1.2}$$

for $\mathbb{Q}$-Brownian motion Z_t independent of W_t, but more general settings are allowed.

Let $X_t = \log(S_t/S_0)$. The *realised quadratic variation* (or *realised variance*) of X_t in $[0,T]$ is

$$\langle X \rangle_T = \int_0^T \sigma_t^2 dt. \tag{1.3}$$

The *realised volatility* will thus be $\langle X \rangle_T^{\frac{1}{2}}$.

Definition 1.1. Let $\overline{\sigma} > 0$ be a constant. A *target volatility call option* is the contingent claim on S_t and $\langle X \rangle_t$ paying

$$H(S_T, K, \langle X \rangle_T) = \overline{\sigma} \left(\frac{T}{\langle X \rangle_T} \right)^{\frac{1}{2}} (S_T - K)^+ \tag{1.4}$$

at time T. The constant $\overline{\sigma}$ is called the *target volatility.*

A target volatility put option is defined in a similar fashion. Note that as T tends to zero H remains well defined by the mean integral Theorem and tends to the intrinsic value $\overline{\sigma}(S_0 - K)^+/\sigma_0$.

The contract we are studying is a kind of European type call option, which pays at maturity a proportion of the vanilla European call on the ratio of the average volatility realised during the life of the contract, and the target volatility, our "bet" volatility level. The aim of this paper is to provide some analytical and semi-analytical formulas for the time-0 value of (1.4) in the setting just presented. According to the risk-neutral pricing formula this amounts to computing

$$\mathbb{E}[H(S_T, K, \langle X \rangle_T)] = \mathbb{E}\left[\overline{\sigma}\left(\frac{T}{\langle X \rangle_T}\right)^{\frac{1}{2}} (S_T - K)^+\right]. \tag{1.5}$$

Clearly in (1.5) and everywhere else the expectation is taken with respect to $\mathbb{Q}$.

We are now going to analyse the equivalent payoff $h(x)$ and its risk-neutral expectation in greater detail.

1.3 Properties of the payoff

The first important remark about (1.4) is that even though it is a joint function of the stock price and variance, it is of the form $p(S_T)q(\langle X \rangle_T)$, for measurable functions p and q. Indeed, when σ_t and W_t are *independent*, a standard conditioning argument ensures that we can write (1.5) in a nicer equivalent way:

Proposition 1.2. *Let σ_t be independent of W_t. Let $C^{BS}(S_0, K, x)$ be the Black-Scholes price of the vanilla european call option of initial underlying value S_0, strike K and total realised variance x on $[0, T]$, that is*

$$C^{BS}(S_0, K, x) = S_0 \mathcal{N}(d^+(x)) - K\mathcal{N}(d^-(x)) \tag{1.6}$$

where $\mathcal{N}(\cdot)$ is the cumulative normal distribution and

$$d^{\pm}(x) = \frac{\log(S_0/K) \pm x/2}{x^{1/2}}. \tag{1.7}$$

Then the function

$$h(x) = \overline{\sigma}\sqrt{T}\frac{C^{BS}(S_0, K, x)}{x^{1/2}} \tag{1.8}$$

is such that

$$\mathbb{E}[H(S_T, K, \langle X \rangle_T)] = \mathbb{E}[h(\langle X \rangle_T)]. \tag{1.9}$$

Proof. Let

$$\mathcal{F}_T^{\sigma} = \{\sigma_t^{-1}(B), B \in \mathcal{B}([0, T]), t \in [0, T]\} \tag{1.10}$$

be the filtration generated by the process σ_t at time T. Conditioning H with respect of $\mathcal{F}_T^{\sigma}$ and taking the expectation we have

$$\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{\langle X \rangle_T^{\frac{1}{2}}}(S_T - K)^+\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{\langle X \rangle_T^{\frac{1}{2}}}(S_T - K)^+ \middle| \mathcal{F}_T^{\sigma}\right]\right]. \tag{1.11}$$

By $\mathcal{F}_T^\sigma$ measurability the factor $\overline{\sigma}/\langle X\rangle_T^{\frac{1}{2}}$ drops out of the inner term. Moreover independence of W_t and $\mathcal{F}_t^\sigma$ implies that conditional on $\mathcal{F}_T^\sigma$, the process W_t is still a Brownian motion, and therefore at each time t, the random variable S_t/S_0 conditional on $\mathcal{F}_T^\sigma$ is log-normal with known instantaneous volatility σ_t (see [11] for a precise account). Therefore we can use the Black-Scholes formula for a call option to conclude

$$\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{\langle X\rangle_T^{\frac{1}{2}}}\mathbb{E}[(S_T-K)^+|\,\mathcal{F}_T^\sigma]\right] = \mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{\langle X\rangle_T^{\frac{1}{2}}}C^{BS}(S_0,K,\langle X\rangle_T)\right] \tag{1.12}$$

$$= \mathbb{E}[h(\langle X\rangle_T)]. \tag{1.13}$$

□

Under independence, the pricing problem has been therefore reduced to the pricing of a claim on the stock quadratic variation only. Extensive treatment of this kind of claims is given in Carr and Lee, [4]. Nevertheless, for $h(x)$ as in (1.8), the results of [4] cannot be directly applied in order to get a useful pricing formula. Indeed depending on the parameters S_0 and K, the function $h(x)$ may or may not be bounded on the half real line, thus not falling under the cases accounted there.

A further issue is that integrability conditions for $h(\langle X\rangle_T)$ may not hold for some ill-behaved processes σ_t, as a consequence of the following Lemma:

Lemma 1.3. *Let $h(x)$ be as in (1.8). Then*

$$\lim_{x\to 0^+} h(x) = \begin{cases} 0 & \textit{if } S_0 < K, \\ \overline{\sigma}\sqrt{T}S_0/\sqrt{2\pi} & \textit{if } S_0 = K, \\ O(x^{-1/2}) & \textit{if } S_0 > K. \end{cases} \tag{1.14}$$

Proof. Let $S_0 < K$. If $x \to 0^+$, $d^+(x)$ and $d^-(x)$ tend both to $-\infty$ and $\mathcal{N}(d^\pm(x)) \to 0$. The asymptotic series for $\mathcal{N}(z)$ as $z \to -\infty$ is

$$\frac{e^{(-z^2/2)}}{\sqrt{2\pi}}(z^{-1} + O(z^{-2})) \tag{1.15}$$

and so as x goes to 0 from the right

$$S_0\frac{\mathcal{N}(d^+(x))}{x^{\frac{1}{2}}} = \frac{S_0}{\sqrt{2\pi}}e^{-\frac{d^+(x)^2}{2}}\left(\frac{1}{\log(S/K)+x/2} + O(x^{1/2})\right) \to 0. \tag{1.16}$$

The same holds for $K\mathcal{N}(d^-(x))/x^{1/2}$, and we have the first line of the claim. If $S_0 = K$ then as $x \to 0^+$ the numerator of $h(x)$ tends to 0 because $\mathcal{N}(d^\pm(x)) = \mathcal{N}(\pm x^{1/2}/2) \to 1$. The McLaurin series for $\mathcal{N}(z)$ in 0 is

$$\mathcal{N}(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}(z + O(z^2)) \tag{1.17}$$

therefore

$$h(x) = \frac{\overline{\sigma}\sqrt{T}S_0}{x^{1/2}}\left(\mathcal{N}(x^{1/2}/2) - \mathcal{N}(-x^{1/2}/2)\right) \tag{1.18}$$

$$= \frac{\overline{\sigma}\sqrt{T}S_0}{x^{1/2}}2\left(\frac{x^{1/2}}{2\sqrt{2\pi}}\right) + O(x) \to \frac{\overline{\sigma}\sqrt{T}S_0}{\sqrt{2\pi}}. \tag{1.19}$$

Finally if $S_0 > K$ then $\mathcal{N}(d^\pm(x)) \to 1$, the numerator remains bounded, and $h(x)$ diverges asymptotically as $x^{-1/2}$. □

The intuition behind this Lemma can be understood by looking directly at (1.4). If the option begins out-of-the-money and the volatility is sufficiently small the payoff will not be triggered, regardless of how big $1/\langle X\rangle_T^{1/2}$ can get. On the other hand if the options begins in-the-money then the difference between terminal stock and strike is going to be positive for small values of volatility, while the inverse square root of the volatility diverges.

From Lemma 1.3 we see that if $S_0 > K$, that is, the option is in-the-money, in principle nothing ensures that the expectation in (1.5) exists. Therefore *extra assumptions must be made on σ_t for the value in (1.5) to be finite.* For the purposes of this paper we will directly assume σ_t to be such that

$$\mathbb{E}[h(\langle X\rangle_T)] < +\infty. \tag{\textbf{B}}$$

For instance this is the case for numerous CIR volatility models with constant parameters κ, θ, η (for the analytic formula of $\langle X\rangle_T$ in the CIR model see [7]). Intuitively for (**B**) not to hold true we should have a degenerate process for the volatility σ_t "falling" in an uncontrolled way around 0 , making (1.4) indefinitely big.

1.4 Fundamental results

Here we develop all the fundamental machinery for our proofs. The main references for this material are the already memtioned paper by Carr and Lee [4], and the book by Gatheral ([10]).

First of all we are going to need the classic result on exponential claims. Since next we are going to be interested in replication (Proposition 1.5) we give the full proof for time-t

values of the payoff. Anyway, in the proofs throughout this paper we only make use of the case $t = 0$.

Proposition 1.4. *Let S_t and X_t as per our hypotheses and assume σ_t is independent of W_t. For all $\lambda \in \mathbb{C}$ the following relation holds:*

$$\mathbb{E}_t[e^{\lambda(\langle X\rangle_T - \langle X\rangle_t)}] = \mathbb{E}_t[e^{p^{\pm}(\lambda)(X_T - X_t)}], \tag{1.20}$$

with

$$p^{\pm}(\lambda) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}. \tag{1.21}$$

In particular $\mathbb{E}[e^{p^{\pm}(\lambda)(X_T - X_t)}] < +\infty$, for all $\lambda \in \mathbb{C}$.

Proof. As mentioned in Proposition 1.2, due to the independence of σ_t and W_t, the conditional distribution of $X_T - X_t$ given $\mathcal{F}_T^{\sigma}$ is normal with mean $-(\langle X\rangle_T - \langle X\rangle_t)/2$ and variance $\langle X\rangle_T - \langle X\rangle_t$. Hence, for all $p \in \mathbb{C}$, if $\mathcal{F}_t^W$ is the filtration generated by W_t

$$\mathbb{E}_t\left[e^{p(X_T - X_t)}\right] = \mathbb{E}_t\left[\mathbb{E}\left[e^{p(X_T - X_t)}|\sigma(F_T^{\sigma}, \mathcal{F}_t^W)\right]\right] = \mathbb{E}_t\left[e^{\left(\frac{p^2}{2} - \frac{p}{2}\right)(\langle X\rangle_T - \langle X\rangle_t)}\right] \tag{1.22}$$

$$= \mathbb{E}_t\left[e^{\lambda(\langle X\rangle_T - \langle X\rangle_t)}\right]. \tag{1.23}$$

having set $\lambda = p^2/2 - p/2$. We deduce that if $\lambda \in \mathbb{C}$ is given, by choosing p to be any of the roots of this equation (1.20) holds. The second claim is a direct consequence of (**A**).

□

Remark 1.1. If for $\lambda \in \mathbb{R}$ and $t = 0$ we denote by $\mathcal{L}\langle X\rangle_T(\lambda)$ the Laplace transform of the distribution of $\langle X\rangle_T$ and by $\mathcal{L}X_T(\lambda)$ the Laplace transform of the distribution of X_T. Proposition 1.4 says that

$$\mathcal{L}\langle X\rangle_T(-\lambda) = \mathcal{L}X_T(-p^{\pm}(-\lambda)). \tag{1.24}$$

This means that any exponential claim $f(\langle X\rangle_T) = e^{\lambda\langle X\rangle_T}$ on the quadratic variation can be priced directly if an analytic closed formula for the Laplace transform of X_T is known. Conversely the distribution function of $\langle X\rangle_T$ is completely determined by the time t prices of the claim $(S_T/S_0)^{p^{\pm}(\lambda)}$, which in turn, by Proposition 1.6 below, can be determined exactly by looking at the spot prices of the European put and call options.

Remark 1.2. As stressed in [4], perhaps an even more interesting reading of Proposition 1.4 is that it effectively allows one to reduce any exponential claim $e^{\lambda\langle X\rangle_T}$ on the quadratic variation $\langle X\rangle_T$ to the power payoff $g(S_T) = (S_T/S_0)^{p^{\pm}(\lambda)}$ on S_T. That is, the *pricing of the path dependent claim f has been reduced to pricing the function g of the terminal stock price alone*. Even though the distribution of S_T is not known, pricing of $g(S_T)$ is always possible by replicating S_T via Proposition 1.6.

This approach to pricing is both *robust* and *non-parametric*. By robust it is meant that it is valid for all stock price process independently of the class of the processes σ_t assumed for the volatility (local volatility, full stochastic volatility, Lévy processes); by non-parametric that (under our assumptions) *we do not need* to specify and calibrate the parameters driving σ_t.

For exponential claims one can also exhibit an explicit *dynamic* hedging strategy for the derivative. We have the following:

Proposition 1.5. *Under the assumptions of Proposition 1.4 the claim $e^{\lambda\langle X\rangle_T}$ is replicated by the self-financing portfolio consisting of*

$$\begin{aligned} &\frac{e^{\lambda\langle X\rangle_t}}{S_t^p} \quad \textit{p-power claims on } S_t, \\ &\frac{-pe^{\lambda\langle X\rangle_t}\mathbb{E}_t[S_T^p]}{S_t^{p+1}} \quad \textit{shares, and} \qquad (1.25) \\ &\frac{pe^{\lambda\langle X\rangle_t}}{S_t^p}\mathbb{E}_t[S_T^p] \quad \textit{cash.} \end{aligned}$$

Proof. At each time t we have the value

$$\frac{e^{\lambda\langle X\rangle_t}}{S_t^p}\mathbb{E}_t[S_T^p] - pe^{\lambda\langle X\rangle_t}\frac{\mathbb{E}_t[S_T^p]}{S_t^{p+1}}S_t + \frac{pe^{\lambda\langle X\rangle_t}\mathbb{E}_t[S_T^p]}{S_t^p} = \frac{e^{\lambda\langle X\rangle_t}\mathbb{E}_t[S_T^p]}{S_t^p} \qquad (1.26)$$

$$= \mathbb{E}_t[e^{\lambda\langle X\rangle_T}], \qquad (1.27)$$

by use Proposition 1.4 in the last equality. Therefore the portfolio replicates the value of the option. To see the self-financing condition set $P_t = e^{\lambda\langle X\rangle_t}/S_t^p$ and $V_t = \mathbb{E}_t[S_T^p]$. By means of the product rule and since $dP_t = -pV_t dS_t/S_t$ we have

$$d(P_tV_t) = P_t dV_t + V_t dP_t + d\langle P, V\rangle_t \qquad (1.28)$$

$$= P_t dV_t - \frac{pP_tV_t}{S_t}dS_t + d\langle P, V\rangle_t. \qquad (1.29)$$

where $d\langle P, V\rangle_t$ has finite variation. But now $P_tV_t = \mathbb{E}_t[e^{\lambda\langle X\rangle_T}]$ is a martingale, and the stochastic integrals in dV_t and in dS_t are local martingales because both V_t and S_t are martingales. The only possibility is then $d\langle P, V\rangle_t = 0$. Since there is no market drift, the cash account B_t is the constant process and $dB_t = 0$. Therefore

$$d(P_tV_t) = P_t dV_t - \frac{pP_tV_t}{S_t} dS_t \tag{1.30}$$

$$= P_t dV_t - \frac{pP_tV_t}{S_t} dS_t + pV_tP_t dB_t. \tag{1.31}$$

The portfolio instant change is only due to change in value of the p-power claim, of the underlying, and of the money account. □

Before moving on we remind the pricing Theorem of decomposition into put and calls cited in Remark 1.2, which first appeared in [3].

Proposition 1.6. *Let S_t be as in our assumptions, and $G \in C^2([0,\infty),\mathbb{R})$. Assume European call and put Options of all strikes are traded for each maturity date T. Denote $P^{BS}(S_0, K)$ and $C^{BS}(S_0, K)$ the Black-Scholes values respectively of vanilla European puts/calls of strike K and initial spot price S_0. Then*

$$\mathbb{E}[G(S_T)] = G(S_0) + \int_0^{S_0} P^{BS}(S_0, K)G''(K)dK + \int_{S_0}^{\infty} C^{BS}(S_0, K)G''(K)dK. \tag{1.32}$$

That is to say, every claim on S_T can be replicated as a continuous portfolio of puts, puts and *calls, or calls (depending upon $S_0 > K$, $S_0 = K$, or $S_0 < K$) of all possible strikes K, each of position $G''(K)$, plus a fixed position of $G(S_0)$ in cash.*

Proofs can be found in [6] and [10]. The assumption that K can tend to infinity is justified by the fact that in practice options can be traded at a sufficiently large number of strikes for (1.6) to represent a good approximation for the real price.

2 At-the-money Taylor expansion

2.1 Motivation

In this Section we show that in a first approximation the at-the-money price of a target volatility option price is basically that of an European option of constant volatility $\overline{\sigma}$. We then move on to study changes in value of the option for strike oscillations close to S_0, by using a Taylor series expansion of the Black-Scholes formula.

Let us consider the at the money value $H(S_0, S_0, \langle X\rangle_T)$ from (1.4). Via (1.9) we can write its initiation value as

$$\mathbb{E}[h(\langle X\rangle_T)] = \mathbb{E}\left[\overline{\sigma}\sqrt{T}\frac{C^{BS}(S_0, S_0, \langle X\rangle_T)}{\langle X\rangle_T^{\frac{1}{2}}}\right] \tag{2.1}$$

where $C^{BS}(S_0, S_0, \langle X\rangle_T)$ is the at-the-money Black-Scholes value of an European Call of volatility $\langle X\rangle_T^{\frac{1}{2}}$. A very accurate estimate of such a value is given by the well known Bachelier approximation (recall that $\langle X\rangle_T^{\frac{1}{2}}$ is the total volatility accumulated during the process)

$$C^{BS}(S_0, S_0, \langle X\rangle_T) \approx S_0\sqrt{\frac{\langle X\rangle_T}{2\pi}} \tag{2.2}$$

which reduces (1.9) to

$$\mathbb{E}[h(\langle X\rangle_T)] \approx S_0\overline{\sigma}\sqrt{\frac{T}{2\pi}}. \tag{2.3}$$

Again, the Bachelier estimate again sets (2.3) to be roughly $C^{BS}(S_0, S_0, \overline{\sigma}^2 T)$.

This comes down to one of the main motivations for the option we are studying. *A target volatility contract is a financial product whose at-the-money value is closely priced by a simple Black-Scholes valuation of volatility $\overline{\sigma}$, however uncertain the market volatility scenario may be.*

The chosen volatility parameter $\overline{\sigma}$, once fed in the Black-Scholes formula, has therefore the effect of setting the level of the price the counterparty is willing to pay for an at-the-money option.

Next, we want to move away from S_0 and explore the change in the pricing of this option for small movements of K around S_0. In doing so is just natural considering the Taylor expansion of the claim (1.8) in K, which only involves the expansion of the Black-Scholes formula appearing in the numerator.

A remarkable feature of such expansion is that, as a function of variance, it consists of a sum of terms each involving an exponential function multiplying an inverse root. If we center it around the at the money point S_0 the dependence on the further factor $\log(S_0/K)$ disappears and we are left with just a sequence of claims depending only on the realised variance.

Hence the approach will be that of expressing the payoff h as a limit of a converging sequence h^ϵ, and then expand in the Taylor power series around S_0 every element of this sequence. The coefficients of the series will turn out to be single claims on the variance which we can replicate exactly by claims on S_T: this is to say that for any claim f on the variance there exists a claim F such that

$$\mathbb{E}[f(\langle X\rangle_T)] = \mathbb{E}[F(S_T)]. \tag{2.4}$$

Exactly as explained in Remark 1.1 for the exponentials, the value of the latter is then by Proposition 1.6 determined uniquely by the time-0 prices of a portfolio of vanilla put and call options.

2.2 Pricing the option

We prove now the pricing result as a limit of a converging sequence of prices. The symbol $\equiv$ stands for equivalence modulo 2: $k \equiv n$ means then that k has same parity as n.

Theorem 2.1 (At-the-money TVO pricing). *Let $H(S_T, K, \langle X\rangle_T)$ be the payoff of the a target volatility call option and let $h(x)$ be defined in (1.9). For $\epsilon > 0$ set H_0^ϵ to be the at-the-money value of the claim $h^\epsilon(\langle X\rangle_T, K)$ with*

$$h^\epsilon(x, K) = h(x+\epsilon, K) = \overline{\sigma}\sqrt{T}\frac{C^{BS}(S_0, K, x+\epsilon)}{(x+\epsilon)^{1/2}}. \tag{2.5}$$

Let $r > 0$ be the radius of analicity of $h(x, K)$ in K. Then for all K such that $|K - S_0| < r$ we have

$$\mathbb{E}[H(S_T, K, \langle X\rangle_T)] = \lim_{\epsilon\to 0} \mathcal{H}^\epsilon(S_T/S_0) \tag{2.6}$$

where

$$\mathcal{H}^\epsilon(S_T, S_0) = H_0^\epsilon + \frac{\overline{\sigma}\sqrt{T}}{\sqrt{2\pi}}\Bigg(\mathbb{E}\left[F^\epsilon(S_T, S_0)\right](K - S_0) + \\ + \sum_{n=0}^{\infty} \frac{(-1)^n (K-S_0)^{n+2}}{S_0^{n+1}(n+2)!} \sum_{k\equiv n}^{n} c_{k,n}\mathbb{E}\left[G^\epsilon_{-\frac{1}{8},(n+2-k)/2}(S_T, S_0)\right]\Bigg) \tag{2.7}$$

for some constants $c_{k,n}$, with

$$F^\epsilon(x, y) = -\frac{1}{\sqrt{2}}\int_0^\infty (x/y)^{p^\pm(-z-1/8)}\frac{e^{-\epsilon(z+1/8)}}{(z+1/8)^{1/2}}dz, \tag{2.8}$$

and, for $s \in \mathbb{R}$ and $t > 0$

$$G^\epsilon_{q,r}(x, y) = \frac{1}{r\Gamma(r)}\int_0^\infty (x/y)^{p^\pm(q+\epsilon-z^{1/r})}e^{-\epsilon z^{1/r}}dz. \tag{2.9}$$

The exponents $p^\pm$ are defined in Proposition 1.4 and the series in (2.7) converges uniformly in K.

Remark 2.1. The number $\mathcal{H}^\epsilon(S_T, S_0)$ is nothing but an equivalent expression of the expectation of the at-the-money Taylor expansion of $h^\epsilon(x, K)$. Indeed this is represented by a convergent series of the values of the claims $F^\epsilon(x, y)$ and $G^\epsilon_{q,r}(x, y)$ on S_T and S_0 all replicating the appropriate functions of $\langle X\rangle_T$ in the Taylor expansion.

We conclude that the time-0 value of a target volatility option is completely determine by S_T and S_0. Moreover by effect of Proposition 1.6, F^ϵ and $G^\epsilon_{q,r}$ can be replicated by an infinite strip of call and put options.

As a Corollary we have an effective approximation of the value of the option.

Corollary 2.2. *Under the assumptions of Theorem (2.1), for $\epsilon > 0$ sufficiently small and $n_0 > 0$ sufficiently large we have the following approximation for the value (1.5)*

$$\mathbb{E}[H(S_T, K, \langle X\rangle_T)] \approx C^{BS}(S_0, S_0, \overline{\sigma}^2 T) + \frac{\overline{\sigma}\sqrt{T}}{\sqrt{2\pi}}\Bigg(\mathbb{E}[F^\epsilon(S_T, S_0)](K - S_0) + \sum_{n=0}^{n_0} \frac{(-1)^n (K - S_0)^{n+2}}{S_0^{n+1}(n+2)!} \sum_{k\equiv n} c_{k,n}\mathbb{E}[G^\epsilon_{-\frac{1}{8},(n+2-k)/2}(S_T, S_0)]\Bigg). \quad (2.10)$$

Proof. Fix $\epsilon > 0$ such that $\mathcal{H}^\epsilon(S_T/S_0)$ is close to the option value as desired, and n_0 such that the partial sums up to n_0 of the series in (2.7) achieves the desired accuracy. Then, exactly as remarked in the previous subsection, applying twice Bachelier formula allows the estimates

$$\mathbb{E}[h^\epsilon(\langle X\rangle_T)] = \mathbb{E}\left[\sigma\sqrt{T}\frac{C^{BS}(S_0, S_0, \langle X\rangle_T + \epsilon)}{\sqrt{\langle X\rangle_T + \epsilon}}\right] \quad (2.11)$$

$$\approx \mathbb{E}\left[\overline{\sigma}\sqrt{T}\frac{S_0}{(\langle X\rangle_T + \epsilon)^{\frac{1}{2}}}\sqrt{\frac{\langle X\rangle_T + \epsilon}{2\pi}}\right] \quad (2.12)$$

$$\approx C^{BS}(S_0, S_0, \overline{\sigma}^2 T). \quad (2.13)$$

and the Corollary then follows by Theorem 2.1. □

We break the proof of Theorem 2.1 in a series of Lemmas and Propositions. In first place we would like to obtain the general formula for the n-th derivative of C^{BS} with respect to the strike. Such a derivative can be expressed as the derivative of a put option value with respect to price, with strike and price inverted. Hence, one has for the first two orders the familiar formulas for the delta and gamma of a put option in the variable K.

$$C'(K) = -\mathcal{N}(d^-(K)) \quad (2.14)$$

and

$$C''(K) = \frac{1}{K\langle X\rangle_T^{\frac{1}{2}}}\phi(d^-(K)). \tag{2.15}$$

Here and everywhere else $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$.

In [8] Estrella derives an explicit formula for the higher order derivatives of a vanilla European call option. This formula applies in our context, because all the derivatives of call and put options of order higher than 1 coincide.

Lemma 2.3. *Let $n \geq 0$. Denote $C^{n+2}(S_0)$ the n+2-th derivative of $C^{BS}(S_0, K, \langle X\rangle_T)$ with respect to K evaluated at S_0. Then*

$$C^{(n+2)}(S_0) = \frac{(-1)^n}{\sqrt{2\pi}S_0^{n+1}}\sum_{k\equiv n} c_{k,n}\frac{e^{-\frac{\langle X\rangle_T}{8}}}{\langle X\rangle_T^{\frac{n+1-k}{2}}}. \tag{2.16}$$

Proof. From [8], setting $\sigma = \langle X\rangle_T^{\frac{1}{2}}$, and $h = d^-(K)$ we have

$$C^{n+2}(K) = \frac{(-1)^n P_n(h)\phi(h)}{(S_0\sigma)^{n+1}} \tag{2.17}$$

for polynomials P_n of degree n satisfying the recursive relation

$$P_n(h) = (h + n\sigma)P_{n-1}(h) - P'_{n-1}(h). \tag{2.18}$$

Since we are developing at-the-money we want to compute this in $K = S_0$, that is, for $h = \sigma/2$. Starting by induction from $P_0 = 1$ it is easy to see from (2.18) that all of the monomials appearing in the polynomial $P_n(\sigma)$ are those having same parity as n. This is because both multiplying $P_{n-1}(\sigma/2)$ by $(2n+1)\sigma/2$ and differentiating it changes the parity of the exponents of such monomials according with that of n. Thus dividing out $P_n(\sigma/2)$ by the factor σ^{n+1} yields a sum of rational functions in σ all having *odd* negative exponent: we then have

$$C^{(n+2)}(S_0) = \frac{(-1)^n}{S_0^{n+1}}\sum_{k\equiv n} c_{k,n}\frac{\sigma^k\phi(\sigma/2)}{\sigma^{n+1}} \tag{2.19}$$

where $c_{k,n}$ are the coefficients given by (2.18). Substituting back $\sigma = \langle X\rangle_T^{\frac{1}{2}}$ and

$$\phi(h) = \exp(-\langle X\rangle_T/8)/\sqrt{2\pi}) \tag{2.20}$$

we see that (2.17) is equivalent to

$$C^{(n+2)}(S_0) = \frac{(-1)^n}{\sqrt{2\pi} S_0^{n+1}} \sum_{k \equiv n} c_{k,n} \frac{e^{-\frac{\langle X \rangle_T}{8}}}{\langle X \rangle_T^{\frac{n+1-k}{2}}}. \tag{2.21}$$

□

Lemma 2.3 provides the explicit decomposition for the payoff of an European call in the series of elementary functions of the variance we are going to need. These claims to appear in Theorem 2.1 "shifted " by ϵ; the next step is then too compute explicitly the risk-neutral expectation of this shifted claims.

Proposition 2.4. *Let $q \in \mathbb{R}$, $r > 0$, $\epsilon > 0$ and*

$$g^{\epsilon}_{q,r}(x) = \frac{e^{q(x+\epsilon)}}{(x+\epsilon)^r}. \tag{2.22}$$

Let $\langle X \rangle_T$ be the realised volatility of S_t at time T. Then

$$\mathbb{E}[g^{\epsilon}_{q,r}(\langle X \rangle_T)] = \mathbb{E}[G^{\epsilon}_{q,r}(S_T, S_0)] \tag{2.23}$$

where

$$G^{\epsilon}_{q,r}(x,y) = \frac{1}{r\Gamma(r)} \int_0^{\infty} (x/y)^{p^{\pm}(q+\epsilon-z^{1/r})} e^{-\epsilon z^{1/r}} dz \tag{2.24}$$

and $p^{\pm}$ is as in Proposition 1.4.

Proof. Let us consider the identity (cf. [2])

$$\frac{1}{a^b} = \frac{1}{a\Gamma(a)} \int_0^{\infty} e^{-bz^{1/r}} dz \qquad a, b > 0. \tag{2.25}$$

By Proposition 1.4, and assuming we can apply Fubini's Theorem we have

$$\mathbb{E}\left[\frac{e^{(q+\epsilon)\langle X \rangle_T}}{(\langle X \rangle_T + \epsilon)^r}\right] = \mathbb{E}\left[\frac{1}{r\Gamma(r)} \int_0^{\infty} e^{(q+\epsilon)\langle X \rangle_T - z^{1/r}(\langle X \rangle_T + \epsilon)} dz\right] \tag{2.26}$$

$$= \frac{1}{r\Gamma(r)} \int_0^{\infty} \mathbb{E}\left[e^{(q+\epsilon-z^{1/r})\langle X \rangle_T}\right] e^{-\epsilon z^{1/r}} dz \tag{2.27}$$

$$= \frac{1}{r\Gamma(r)} \int_0^{\infty} \mathbb{E}\left[\left(\frac{S_T}{S_0}\right)^{p^{\pm}(q+\epsilon-z^{1/r})}\right] e^{-\epsilon z^{1/r}} dz \tag{2.28}$$

$$= \mathbb{E}\left[\frac{1}{r\Gamma(r)} \int_0^{\infty} \left(\frac{S_T}{S_0}\right)^{p^{\pm}(q+\epsilon-z^{1/r})} e^{-\epsilon z^{1/r}} dz\right]. \tag{2.29}$$

Indeed, being σ_t bounded by assumption (**A**), so is $\langle X \rangle_T$, say $\langle X \rangle_T < M$, for $M > 0$. Thus $\forall z > 0$

$$\mathbb{E}\left[e^{(q+\epsilon)\langle X \rangle_T - z^{1/r}\langle X \rangle_T}\right] < \mathbb{E}\left[e^{(q+\epsilon)\langle X \rangle_T}\right] < e^{(q+\epsilon)M}. \tag{2.30}$$

Therefore the integral in (2.27) converges because so does $e^{-\epsilon z^{1/r}}$; this justifies the first application of Fubini. Secondly observe that $p\pm$ as a function of z is a complex exponent of constant norm, which means that $|(S_T/S_0)^{p^{\pm}(q+\epsilon-z^{1/r})}| \leq |S_T/S_0|^c = e^{cX_T}$ for some positive real constant c. Therefore if $C = \int_0^{\infty} e^{-\epsilon z^{1/r}} dz$ then

$$\mathbb{E}\left[\frac{1}{r\Gamma(r)}\int_0^{\infty}\left(\frac{S_T}{S_0}\right)^{p^{\pm}(q+\epsilon-z^{1/r})} e^{-\epsilon z^{1/r}}dz\right] \leq \mathbb{E}\left[\frac{1}{r\Gamma(r)}e^{cX_T}C\right] \tag{2.31}$$

and the last quantity is finite by Proposition 1.4. Having proved the convergence of this integral, the last equality is a fortiori established and this yields the result.

□

Combined with Lemma 2.3, Proposition 2.4 effectively reduces in the form (2.4) the claims on the variance of order equal or greater than two, but leaves out the first order. This is precisely what the next Proposition takes care of:

Proposition 2.5. *Let $\epsilon > 0$ and*

$$f^{\epsilon}(x) = -\sqrt{2\pi}\frac{\mathcal{N}(-\sqrt{x+\epsilon})}{\sqrt{x+\epsilon}}. \tag{2.32}$$

Then

$$\mathbb{E}[f^{\epsilon}(\langle X\rangle_T)] = \mathbb{E}[F^{\epsilon}(S_T, S_0)] \tag{2.33}$$

where F^{ϵ} is that defined in equation (2.8)

Proof. Let $t(z) = 1/\sqrt{\pi(z+1/8)}$. It is easy to verify that the Laplace transform of $t(z)$

$$\mathcal{L}t(x) = \frac{e^{x/8}}{\sqrt{x}}\text{erfc}(\sqrt{x/8}) \tag{2.34}$$

where

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}du. \tag{2.35}$$

Now clearly we have the relations

$$\mathcal{N}(-\sqrt{x}/2) = 1 - \mathcal{N}(\sqrt{x}/2) = \frac{1}{2}\text{erfc}(\sqrt{x/8}). \tag{2.36}$$

Multiplying both sides of (2.36) by $e^{x/8}/\sqrt{x}$ and using (2.34) yields

$$\frac{\mathcal{N}(-\sqrt{x}/2)e^{x/8}}{\sqrt{x}} = \frac{1}{2}\mathcal{L}t(x) \tag{2.37}$$

that is

$$\frac{\mathcal{N}(-\sqrt{x}/2)}{\sqrt{x}} = \frac{e^{-x/8}}{2}\mathcal{L}t(x) = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{e^{(-z-1/8)x}}{(z+1/8)^{1/2}}dz. \tag{2.38}$$

Therefore if we set $x = \langle X\rangle_T + \epsilon$ we obtain

$$\frac{\mathcal{N}(-\sqrt{\langle X\rangle_T+\epsilon}/2)}{\sqrt{\langle X\rangle_T+\epsilon}} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{e^{(-z-1/8)\langle X\rangle_T}e^{-\epsilon(z+1/8)}}{(z+1/8)^{1/2}}dz. \tag{2.39}$$

When taking the expectation the usual Fubini argument applies because by assumption (**A**) $\mathbb{E}\left[e^{(-z-1/8)\langle X\rangle_T}\right]$ is bounded and the non-random part is integrable in z. After applying Proposition 1.4 we can take the expectation out of the integral because $(S_T/S_0)^{p^{\pm}(-z-1/8)}$ is bounded and $e^{-\epsilon(z+1/8)}/\sqrt{z+1/8}$ integrable. This means

$$\mathbb{E}\left[\frac{\mathcal{N}(-\sqrt{\langle X\rangle_T+\epsilon}/2)}{\sqrt{\langle X\rangle_T+\epsilon}}\right] = \frac{1}{2\sqrt{\pi}}\int_0^\infty \mathbb{E}\left[(S_T/S_0)^{p^{\pm}(-z-1/8)}\right]\frac{e^{-\epsilon(z+1/8)}}{(z+1/8)^{1/2}}dz \tag{2.40}$$

$$= \frac{1}{2\sqrt{\pi}}\mathbb{E}\left[\int_0^\infty \frac{(S_T/S_0)^{p^{\pm}(-z-1/8)}e^{-\epsilon(z+1/8)}}{(z+1/8)^{1/2}}\right]dz \tag{2.41}$$

and the statement follows by multiplication of both sides by $-\sqrt{2\pi}$. □

We are now ready for the proof of the main Theorem.

Proof of Theorem 2.1. By continuity of h we have

$$\lim_{\epsilon\to 0} h^\epsilon(\langle X\rangle_T, K) = h(\langle X\rangle_T, K) \tag{2.42}$$

almost surely; the random variables on the left side are all $L^1(\mathbb{Q})$, and so is the right side by assumption (**B**). It is easy to see that depending upon $K \le S_0$ or $K > S_0$ the function $h^\epsilon(x, K)$ is either monotone increasing or decreasing in x; therefore by the Dominated and Monotone convergence Theorems, and Proposition 1.2

$$\lim_{\epsilon\to 0}\mathbb{E}[h^\epsilon(\langle X\rangle_T)] = \mathbb{E}[h(\langle X\rangle_T)] = \mathbb{E}[H(S_T, K, \langle X\rangle_T)]. \tag{2.43}$$

If $r > 0$ is the radius of analicity of the Black Scholes formula in K, we develop $h^\epsilon(x, K)$ as a function of K in its Taylor series around S_0. By virtue of (2.14) and Lemma 2.3, we have, for all $x > 0$:

$$\begin{aligned} h^\epsilon(x,K) = h_0^\epsilon + \frac{\overline{\sigma}\sqrt{T}}{\sqrt{x+\epsilon}}\Bigg(& -\mathcal{N}(-\sqrt{x+\epsilon})(K-S_0) \\ & + \sum_{n=0}^{\infty} \frac{(-1)^n (K-S_0)^{(n+2)}}{(n+2)!\sqrt{2\pi}S_0^{n+1}} \sum_{k\equiv n} c_{k,n} \frac{e^{-\frac{x+\epsilon}{8}}}{(x+\epsilon)^{\frac{n+1-k}{2}}} \Bigg) \end{aligned} \tag{2.44}$$

$$= h_0^\epsilon + \overline{\sigma}\sqrt{T}\left(\frac{q^\epsilon(x)}{\sqrt{2\pi}}(K-S_0) + \sum_{n=0}^{\infty} \frac{(-1)^n (K-S_0)^{(n+2)}}{(n+2)!\sqrt{2\pi}S_0^{n+1}} \sum_{k\equiv n} c_{k,n} g^\epsilon_{-1/8,(n+2-k)/2}(x) \right). \tag{2.45}$$

where $h_0^\epsilon = h^\epsilon(x, S_0)$. Then we calculate (2.45) in $x = \langle X \rangle_T$ and take the expectation. Being every random variable involved $L^1(\mathbb{Q})$ it drops into the series; the proof is then complete by applying Propositions 2.4 and 2.5.

□

Remark 2.2. If we have an explicit analytical formula for the Laplace transform of $\langle X \rangle_T$ which is also integrable, then we may find the pricing formula simply by computing integrals in the form

$$\frac{1}{r\Gamma(r)} \int_0^\infty \mathbb{E}\left[e^{(q - z^{1/r})\langle X \rangle_T} \right] dz \tag{2.46}$$

and

$$-\frac{1}{\sqrt{2}} \int_0^\infty \frac{\mathbb{E}\left[e^{(-z-1/8)\langle X \rangle_T} \right]}{(z+1/8)^{1/2}} dz \tag{2.47}$$

These are analogous to those of Propositions 2.4 and 2.5 but are deduced from the coefficients of the Taylor power series in K around S_0 *of the function h(x) itself.* The integrability of the Laplace transform of the distibution of $\langle X \rangle_T$ indeed provides a sufficient condition for direct manipulation of the payoff, without relying on a convergence argument. This is the setting in which we numerically implemented our results (Section 5).

3 Log-strike Fourier and Laplace transform method

We will now derive exact formulas for both the Fourier and the Laplace transform for the TVO price, as expressed in the log-strike price variable. Once we have these, the value of the option is then obtained by numerically inverting the transform and undoing the variable change.

As opposed to what has been performed in Section 2, the formulas that we will obtain are *exact*; nevertheless a numerical inversion of the Fourier transform is needed to determine the actual option value.

We are going to calculate the transforms via the same methodology involved by Proposition 1.4 which has been used in the previous part. Anyway the final form for the transforms of the option value we are going to show requires evaluation of an expectation of an *infinite* one parameter-family of claims, one for each parameter.

The formal arguments underlying the two derivations are indeed very similar. We are presenting both because they have been designed for different numerical inversion algorithms. Inversion of the Fourier method is to be performed with the classic FFT inversion of [5] , whereas an excellent algorithm for the Laplace inversion is that of Abate-Whitt, [1].

3.1 The Fourier transform

In this subsection we will obtain our own formula for the inversion by closely following the steps of [5]. As was the case in Section 2 for technical reasons of integrability, in general we cannot work directly on the payoff defined in (1.4) but instead produce a series of claim converging to H in $L^1(\mathbb{Q})$.

For $\epsilon \geq 0$ call

$$V^\epsilon(K) = \mathbb{E}[H(S_T, K, \langle X\rangle_T + \epsilon)]. \tag{3.1}$$

So that $V0$ is the value of the option. Set $\kappa = \log K$ and $Y_T = \log S_T$. The first important remark to do is that $V^\epsilon(K)$ is not directly integrable as a function of the log strike. In order to achieve (square) integrability we must work instead on a modified version of the payoff by choosing parameter $\alpha > 0$ and multiply V^ϵ by the dampening factor $e^{\alpha\kappa}$ so to define the dampened log-strike value of $V^\epsilon(K)$ as

$$v^\epsilon_\alpha(\kappa) := e^{\alpha\kappa} V^\epsilon(e^\kappa) = \overline{\sigma}\sqrt{T}\mathbb{E}\left[\frac{e^{\alpha\kappa}}{(\langle X\rangle_T + \epsilon)^{\frac{1}{2}}}(e^{Y_T} - e^\kappa)^+\right] \tag{3.2}$$

We first calculate the Fourier transform, square integrability of $v^\epsilon_\alpha(\kappa)$ will follow.

Proposition 3.1. *Let $v^\epsilon_\alpha(\kappa)$ be as in (3.2). Then its Fourier transform*

$$\hat{v}^\epsilon_\alpha(t) = \int_{-\infty}^{\infty} e^{it\kappa} v^\epsilon_\alpha(\kappa) d\kappa \tag{3.3}$$

satisfies, $\forall t \in \mathbb{R}$.

$$\hat{v}^\epsilon_\alpha(t) = \frac{2S_0^{(\alpha+it+1)}\overline{\sigma}\sqrt{T}}{(\alpha+it+1)(\alpha+it)\sqrt{\pi}} \int_0^\infty \mathbb{E}\left[\left(\frac{x}{y}\right)^{p\pm(\lambda_t(z))}\right] e^{-\epsilon z^2} dz \tag{3.4}$$

where

$$\lambda_t(z) = -z^2 + \frac{1}{2}\left((\alpha+it+1)^2 - (\alpha+it+1)\right). \tag{3.5}$$

Proof. Assuming we can use Fubini's Theorem

$$\hat{v}_\alpha(t) = \int_{-\infty}^{\infty} e^{it\kappa}\ \overline{\sigma}\sqrt{T}\mathbb{E}\left[\frac{e^{\alpha\kappa}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}(e^{Y_T}-e^{\kappa})\mathbb{1}_{\{\kappa<Y_T\}}\right]d\kappa \tag{3.6}$$

$$=\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\int_{-\infty}^{Y_T} e^{\kappa(\alpha+it)+Y_T}-e^{\kappa(\alpha+it+1)}d\kappa\right] \tag{3.7}$$

$$=\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\left(\frac{e^{Y_T(\alpha+it+1)}}{\alpha+it}-\frac{e^{Y_T(\alpha+it+1)}}{\alpha+it+1}\right)\right] \tag{3.8}$$

$$=\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\frac{e^{Y_T(\alpha+it+1)}}{(\alpha+it+1)(\alpha+it)}\right]. \tag{3.9}$$

By conditioning (3.9) with respect to $\mathcal{F}_T^\sigma$ we can disentangle the variables Y_T and $\langle X\rangle_T$ and express the term in the expectation as a function of the quadratic variation alone. Indeed being $\langle X\rangle_T$ measurable with respect of $\mathcal{F}_T^\sigma$ and Y_T independent of it

$$\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\frac{e^{Y_T(\alpha+it+1)}}{(\alpha+it+1)(\alpha+it)}\right] \tag{3.10}$$

$$=\mathbb{E}\left[\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\frac{e^{Y_T(\alpha+it+1)}}{(\alpha+it+1)(\alpha+it)}\middle|\mathcal{F}_T^\sigma\right]\right] \tag{3.11}$$

$$=\mathbb{E}\left[\frac{S_0^{(\alpha+it+1)}\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\mathbb{E}\left[\frac{e^{X_T(\alpha+it+1)}}{(\alpha+it+1)(\alpha+it)}\middle|\mathcal{F}_T^\sigma\right]\right] \tag{3.12}$$

$$=\frac{S_0^{(\alpha+it+1)}\overline{\sigma}\sqrt{T}}{(\alpha+it+1)(\alpha+it)}\mathbb{E}\left[\frac{e^{\eta_\alpha^t\langle X\rangle_T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\right]. \tag{3.13}$$

In the last equality has again been used the fact that S_t is log-normal under conditioning by the filtration generated by σ_t. Moreover, we have set

$$\eta_\alpha^t = \frac{1}{2}\left((\alpha+it+1)^2-(\alpha+it+1)\right). \tag{3.14}$$

As in Proposition 2.1 we can re-write the term under the expectation in (3.13) by using the inverse-root integral representation (2.25) with $b=1/2$. Fubini's Theorem and Proposition 1.4 then yield

$$\hat{v}_\alpha(t) = \frac{2S_0^{(\alpha+it+1)}\overline{\sigma}\sqrt{T}}{(\alpha+it+1)(\alpha+it)\sqrt{\pi}}\int_0^\infty \mathbb{E}\left[e^{-z^2(\langle X\rangle_T+\epsilon)+\eta_\alpha^t\langle X\rangle_T}\right]dz \tag{3.15}$$

$$=\frac{2S_0^{(\alpha+it+1)}\overline{\sigma}\sqrt{T}}{(\alpha+it+1)(\alpha+it)\sqrt{\pi}}\int_0^\infty \mathbb{E}\left[\left(\frac{S_T}{S_0}\right)^{p\pm(\lambda_t(z))}\right]e^{-\epsilon z^2}dz \tag{3.16}$$

where $p^{\pm}(z)$ is that defined in Proposition 1.4 and $\lambda_t(z) = -z^2 + \eta_\alpha^t$. All of the applications of Fubini are then justified because of (**A**) and the integrability of $e^{-\epsilon z^2}$, and this completes the proof. □

As a consequence, by inversion we have a pricing result for the claim $V(K)$:

Corollary 3.2. *The value of the claim $V^0(K)$ of a target volatility option can be computed as*

$$\lim_{\epsilon \to 0} V^\epsilon(K) = V^0(K) \tag{3.17}$$

where

$$V^\epsilon(K) = \frac{1}{2\pi K^\alpha} \int_{-\infty}^{\infty} e^{-it \log K} \hat{v}_\alpha^\epsilon(t) dt \tag{3.18}$$

$$= \frac{1}{\pi K^\alpha} \int_0^\infty e^{-it \log K} \hat{v}_\alpha^\epsilon(t) dt. \tag{3.19}$$

Proof. From (3.16) we immediately see (as usual by Proposition 1.4 and assumption (**A**) that the integral part of the transform stays bounded, while the denominator is $o(t^2)$ as $t \to \pm\infty$. This shows that $\hat{v}_\alpha^\epsilon(t) \in L^2$, which implies that $v_\alpha^\epsilon(\kappa)$ is L^2 as well, so the first equality follows by applying the inversion Theorem and setting back $K = e^\kappa$. The second is because being $V^\epsilon(K)$ real, the function $\hat{v}_\alpha(t)$ must be odd in his imaginary part and even in its real part, so we can write the integral as twice the integral on the half real line.

Observe finally that as $\epsilon \to 0$, the function $V^\epsilon(K)$ tends increasingly to $V(K)$ hence by Monotone convergence it is $v_\alpha^\epsilon(\kappa) \to v_\alpha^\epsilon(\kappa)$ which directly implies 3.17. □

From all we have just seen we have one free parameter $\alpha > 0$ which is not given by the problem, and can be chosen as we want. As explained in [5] setting this parameter in a sensible way is crucial to obtain accuracy and efficiency in the inversion.

We now illustrate the very similar Laplace transform method.

3.2 The Laplace transform

Instead of considering the Fourier transform of a modified payoff of the call option one can think of directly performing the Laplace transform of H in the log-strike. This is not going to be possible because a call TVO, exactly like a vanilla call option, is not integrable in $\log K$. What we are going to do is then the following: first we do the Laplace transform of a TVO *put* option, invert it, and then work back the value of the call option via call/put parity. But first we need to make clear what we mean by put call/parity with and what is a forward contract on the inverse realised volatility:

Definition/Proposition 3.1. *A target volatility forward of strike K on an underlying asset S_t is the contract paying at time T*

$$F_{wd}(S_T, K, \langle X\rangle_T) = \sigma \left(\frac{T}{\langle X\rangle_T}\right)^{\frac{1}{2}} (S_0 - K) \tag{3.20}$$

and its time-0 value is

$$F^0_{wd}(K) = \overline{\sigma}\sqrt{T}(S_0 - K)\mathbb{E}\left[\frac{1}{\langle X\rangle_T^{\frac{1}{2}}}\right]. \tag{3.21}$$

Proof. Taking the risk-neutral expectation in (3.20) and conditioning by $\mathcal{F}^\sigma_T$ we have

$$F^0_{wd}(K) = \overline{\sigma}\sqrt{T}\mathbb{E}\left[\frac{1}{\langle X\rangle_T^{\frac{1}{2}}}\mathbb{E}[S_T - K|\mathcal{F}^\sigma_T]\right] \tag{3.22}$$

$$= \overline{\sigma}\sqrt{T}(S_0 - K)\mathbb{E}\left[\frac{1}{\langle X\rangle_T^{\frac{1}{2}}}\right] \tag{3.23}$$

and the claim then follows by Proposition 2.4. □

Remark 3.1. By Proposition 2.4 we now exactly the value of F^0_{wd}.

It is now immediate to see that the put/call parity for vanilla options implies a put call/parity for TVOs, that is, if $P(K)$ is the value of a put TVO and $V(K)$ the value or a call TVO then

$$V(K) - P(K) = F_{wd}(K). \tag{3.24}$$

Also, as in Proposition 3.1 we have an expression for the Laplace Transform for the modified TVO put value in the log-strike.

Proposition 3.3. *Let $P(K)$ be the value of a TVO put option in the strike K and $\epsilon > 0$. As (3.2) define, for $Y_t = \log S_t$, $K = \log k$, and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 1$*

$$p^\epsilon(\kappa) = P^\epsilon(e^\kappa) = \mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T + \epsilon)^{1/2}}(e^\kappa - e^{Y_T})^+\right]. \tag{3.25}$$

Then

$$\mathcal{L}p^\epsilon(\alpha) = \frac{2S_0^{(1-\alpha)}\overline{\sigma}\sqrt{T}}{(\alpha - 1)\alpha\sqrt{\pi}}\int_0^\infty \mathbb{E}\left[\left(\frac{x}{y}\right)^{p\pm(\lambda_t(z))}\right] e^{-\epsilon z^2} dz. \tag{3.26}$$

where

$$\lambda_t(z) = -z^2 + \frac{1}{2}\left(\alpha^2 - \alpha\right). \tag{3.27}$$

Proof. Firstly, observe that the necessary integrability condition holds true whenever the real part of the Laplace parameter α is bigger than 1, and therefore the Laplace transform $\mathcal{L}^\epsilon p(\alpha)$ of $p^\epsilon(\kappa)$ is well defined for all $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 1$.

The derivation of (3.26) is formally similar to that of equation (3.4) of Proposition 3.1: one integrates then uses the usual conditioning argument to obtain

$$
\begin{aligned}
\mathcal{L}_p^\epsilon(\alpha) =& \int_1^{+\infty} e^{-\alpha\kappa}\, \overline{\sigma}\sqrt{T}\mathbb{E}\left[\frac{1}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}(e^\kappa - e^{Y_T})\mathbb{1}_{\{\kappa > Y_T\}}\right] d\kappa && (3.28)\\
=&\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\int_{Y_T}^{+\infty} e^{\kappa(1-\alpha)} - e^{-\alpha\kappa+Y_T} d\kappa\right] && (3.29)\\
=&\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\left(\frac{e^{Y_T(1-\alpha)}}{1-\alpha} - \frac{e^{Y_T(1-\alpha)}}{\alpha}\right)\right] && (3.30)\\
=&\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\frac{e^{Y_T(1-\alpha)}}{(\alpha-1)\alpha}\right] && (3.31)\\
=&\mathbb{E}\left[\mathbb{E}\left[\frac{\overline{\sigma}\sqrt{T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\frac{e^{Y_T(1-\alpha)}}{(\alpha-1)\alpha}\middle|\mathcal{F}_T^\sigma\right]\right] && (3.32)\\
=&\frac{S_0^{(1-\alpha)}\overline{\sigma}\sqrt{T}}{(\alpha-1)\alpha}\mathbb{E}\left[\frac{e^{\eta_\alpha\langle X\rangle_T}}{(\langle X\rangle_T+\epsilon)^{\frac{1}{2}}}\right] && (3.33)
\end{aligned}
$$

where $\eta_\alpha = \frac{1}{2}(\alpha^2 - \alpha)$. The claim follows again by writing the integral representation of the square root for (3.33) and bringing the expectation inside the resulting integral by means of assumption (**A**). □

Inverting the transform and using the put/call parity we have our result:

Corollary 3.4. *The price $V(K)$ of a TVO call option as dependent of the strike K is given, for all $c > 1$, by*

$$
V(K) = F_{wd}^0(K) + \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha \log K}\mathcal{L}^\epsilon p(\alpha) d\alpha. \qquad (3.34)
$$

Proof. Let us write the put/call parity formula (3.24) for $K = e^\kappa$ and express $p(\kappa)$ as a limit of $p^\epsilon(\kappa)$. If $c > 1$ the function $\mathcal{L}^\epsilon p(\alpha)$ has no poles, and we can then write $p^\epsilon(K)$ as the Bromwhich contour integral of $\mathcal{L}^\epsilon p(\alpha)$. By changing to the variable K the Corollary follows. □

4 Uniform and L^2 convergence

Applying the theory developed in [4], a third way to approach the problem of pricing TVOs is that of writing the equivalent claim (1.8) as a limit in some suitable functional space. An approximation of the price then naturally arises by considering the value of the n-th element of this sequence.

By recalling Lemma 1.3, one sees that this idea is doomed to fail if applied straightforwardly to $h(x)$. This is because if the options begins in-the-money then $h(x)$ is not bounded around 0. However, we can still make use of the idea of modifying the claim a little bit in a way that the new claim does not suffer of the same limitation of $h(x)$, and yet it is sufficiently close to it in value.

The advantage of this kind of approach is that it relies on much simpler and manageable mathematical expressions, and we need not to compute hard integral transforms to find prices, as we did previously. Moreover, an approximate replication of the TVO in a simple portfolio of exponential claims on the variance (or, equivalently, power claims on the stock) will be be possible.

The Banach spaces we are going to consider are both supported by the vector space of continuous functions on the half real line decreasing to 0; one has the topology induced by the uniform norm, the other that of the L^2 norm.

4.1 Bernstein polynomials

Continuous real function on a compact set are known to be uniformly approximated by some sequence of polynomials: this is the Weierstrass Theorem.

Theorem 4.1 (Weierstrass). *Let $f(x)$ be a continuous function on an interval $[a, b]$, $a, b \in \mathbb{R}$. There exists a sequence of polynomials $P_n(x)$ such that $P_n(x) \to f(x)$ uniformly in $[a, b]$.*

To construct explicitly such a sequence sequence typically one makes use of the *Bernstein polynomials.* If in the above we chose $a = 0$, $b = 1$ then we have the more specific

Theorem 4.2 (Bernstein). *Let $f(x)$ be continuous function on $[0, 1]$. The* Bernstein Polynomials *of f of order n in $[a, b]$*

$$B_n f(x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \tag{4.1}$$

are a sequence of polynomials such that $B_n f(x) \to f(x)$ uniformly in $[0, 1]$.

Proof. [12], Theorem 1.1.1. □

Clearly proving the Bernstein Theorem amounts to proving the Weierstrass Theorem because up to a transformation we can always assume we are in the compact $[0,1]$.

All of this being said, it is still unclear why uniform convergence of polynomials could be applied to functions on an unbounded domain, as $h(x)$ is.

Let us consider the Banach space $C([0,1], \| \|_\infty)$ and let

$$\Lambda_0 = \left\{ f : [0,+\infty) \to \mathbb{R}, \text{ such that } \lim_{x\to+\infty} f(x) < +\infty \right\}. \tag{4.2}$$

Then for all $c > 0$ the diffeomorphism

$$\begin{aligned} \psi_c : [0,+\infty) &\rightarrow [0,1] \\ x &\mapsto e^{-cx} \end{aligned} \tag{4.3}$$

pushes back to the linear isomorphism

$$\begin{aligned} \psi_c^* : C([0,1], \| \|_\infty) &\rightarrow \Lambda_0 \\ f(x) &\mapsto f(\psi_c(x)) \end{aligned} \tag{4.4}$$

which naturally induces a Banach space structure on Λ_0 and a corresponding Banach spaces isomorphism. Bernstein Theorem then still holds true in Λ_0; since $h \in \Lambda_0$ this is precisely what we will be using shortly in the main convergence result.

Again, recall that $h(x)$ can be completed to a continuous function on $[0,+\infty)$ if and only if $S_0 \leq K$; therefore we can find a pricing formula in terms of a convergent series of claims for $h^\epsilon(x)$ instead and then let $\epsilon \to 0$.

Proposition 4.3. *Let $h^\epsilon(x)$ be defined (2.5) and let $S_0 \leq K$. We have*

$$\mathbb{E}[h(\langle X\rangle_T)] = \lim_{\epsilon\to 0} \mathbb{E}[h^\epsilon(\langle X\rangle_T)] \tag{4.5}$$

$$\mathbb{E}[h^\epsilon(\langle X\rangle_T)] = \lim_{n\to\infty} \sum_{k=0}^{n} C_k^n \mathbb{E}[P_k(\langle X\rangle_T)] = \sum_{k=0}^{n} C_k^n \mathbb{E}\left[P_k(S_T, S_0)\right] \tag{4.6}$$

with

$$P_k(x,y) = (x/y)^{p^{\pm}(-ck)}, \tag{4.7}$$

$$C_k^n = \sum_{j=1}^{k} (-1)^{k-j} h_*^\epsilon\left(j/n\right) \binom{n}{k} \binom{k}{j} \tag{4.8}$$

and

$$h_*^\epsilon(x) = h^\epsilon\left(-\log x/c\right). \tag{4.9}$$

Proof. The equation (4.5) is clear and has been already proven in Theorem (2.1), so we just need to show (4.6).

Being $h^\epsilon(x)$ continuous on $[0,+\infty)$ and $\lim_{x\to\infty} h^\epsilon(x) = 0$ we see that $h_*^\epsilon(x)$ is uniformly continuous in $[0,1]$ and $h_*^\epsilon(0) = 0$. But then by Bernstein Theorem

$$B_n h_*^\epsilon(y) \to h_*^\epsilon(y) \tag{4.10}$$

uniformly in $[0,1]$. Therefore if $y = e^{-cx}$ equation (4.4) implies that

$$B_n h_*^\epsilon(e^{-x}) \to h_*^\epsilon(e^{-cx}) = h^\epsilon(x) \tag{4.11}$$

uniformly in $[0,+\infty)$. Hence, since $\langle X\rangle_T > 0$ and the payoff h^ϵ is integrable, the Uniform Convergence Theorem yields

$$\mathbb{E}[h^\epsilon(\langle X\rangle_T)] = \mathbb{E}\left[\lim_{n\to\infty} B_n h_*^\epsilon(e^{-c\langle X\rangle_T})\right] = \lim_{n\to\infty} \mathbb{E}\left[B_n h_*^\epsilon(e^{-c\langle X\rangle_T})\right]. \tag{4.12}$$

We must now just compute $\mathbb{E}\left[B_n h_*^\epsilon(e^{-c\langle X\rangle_T})\right]$. Indeed by the Newton binomial formula, shifting the j index, and changing summation order

$$\mathbb{E}\left[B_n h_*^\epsilon(e^{-c\langle X\rangle_T})\right] = \sum_{j=1}^{n} h_*^\epsilon(j/n)\binom{n}{j}\mathbb{E}\left[e^{-cj\langle X\rangle_T}(1-e^{-c\langle X\rangle_T})^{n-j}\right] \tag{4.13}$$

$$= \sum_{j=1}^{n} h_*^\epsilon(j/n)\binom{n}{j}\mathbb{E}\left[e^{-cj\langle X\rangle_T}\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^k e^{-ck\langle X\rangle_T}\right] \tag{4.14}$$

$$= \sum_{j=1}^{n} h_*^\epsilon(j/n)\binom{n}{j}\sum_{k=j}^{n}\binom{n-j}{k-j}(-1)^{k-j}\mathbb{E}\left[e^{-ck\langle X\rangle_T}\right] \tag{4.15}$$

$$= \sum_{k=1}^{n}\sum_{j=1}^{k} h_*^\epsilon(j/n)\binom{n}{j}\binom{n-j}{k-j}(-1)^{k-j}\mathbb{E}\left[e^{-ck\langle X\rangle_T}\right] \tag{4.16}$$

and then (4.6) follows from $\binom{n}{j}\binom{n-j}{k-j} = \binom{n}{k}\binom{k}{j}$ and Proposition 1.4. □

Remark 4.1. Needless to say, all of the above equally applies to the function $h(x)$ directly in the case the options begins at-the-money or out-of-the-money. In this case an explicit approximate hedge can be established for the option. For all fixed n one has just a linear combination of exponential claims; such a portfolio can then be hedged as has been shown in Proposition 1.5.

This Proposition enables an estimate of the option value by truncating the series in (4.6) to the desired n. Acting on c varies the rate on n with which the algorithm converges.

Also, by fixing n and choosing $c < 1/8n$, the approximation provided is given by a real number, and this may be useful in numerical implementation.

Compared to those of previous sections, this calculation is very easy to perform. In fact, we must just compute the n expectations of the claims $e^{-k\langle X\rangle_T}$, the $n(n-1)/2$ binomial coefficients $\binom{n}{k}$, $\binom{k}{j}$ and the n values $h_*^\epsilon(k/n)$.

Uniform convergence from Theorem 4.3 can also be brought into the picture allowing an estimate of the speed of convergence with n of the Bernstein polynomials of $h_*^\epsilon(x)$. We have the following general Proposition:

Proposition 4.4. *Let $f \in C^1([0,1])$. Then if $f'(x) = df/dx$*

$$|f(x) - B_n f(x)| \leq \frac{3}{4} n^{-\frac{1}{2}} \omega_{[0,1]}^{f'}(n^{-\frac{1}{2}}) \tag{4.17}$$

where, for $D \subset \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $\delta > 0$

$$\omega_D^g(\delta) = \sup_{\substack{|x-y|<\delta \\ x,y \in D}} |g(x) - g(y)| \tag{4.18}$$

is the modulus of continuity *of g in D.*

Proof. [12] Theorem 1.6.2. □

We can use this to get a bound for the error between h^ϵ and its approximation in (4.6), thus obtaining an estimate for the rate of convergence to the price of the option.

Proposition 4.5. *Let $h^\epsilon(x)$ be as in (2.5). Assume that $S_0 \leq K$, and let $h_*^\epsilon(x)$ be that of Proposition 4.3. Then there exists $C > 0$ such that*

$$\left| \mathbb{E}[B_n h_*^\epsilon(e^{-c\langle X\rangle_T})] - \mathbb{E}[h^\epsilon(\langle X\rangle_T)] \right| \leq \frac{C}{n}. \tag{4.19}$$

Proof. $(h^\epsilon)'(x) = dh(x)/dx$ is bounded an after the transformation function so it is $(h_\epsilon^*)'(x)$. Let $c = \sup_{[0,1]} |(h_*^\epsilon)'(x)|$. Being $(h_\epsilon^*)'(x)$ differentiable it is in particular Lipschitzian of order 1, and therefore we have

$$\omega_{[0,1]}^{(h_*^\epsilon)'}(\delta)(n^{-1/2}) \leq cn^{-1/2}, \tag{4.20}$$

whence, by Proposition 4.4

$$\left| \mathbb{E}[B_n h_*^\epsilon(e^{-c\langle X\rangle_T})] - \mathbb{E}[h^\epsilon(\langle X\rangle_T)] \right| \leq \mathbb{E}\left[|B_n h_*^\epsilon(e^{-c\langle X\rangle_T}) - h_*^\epsilon(e^{-c\langle X\rangle_T})| \right] \tag{4.21}$$

$$\leq \frac{3}{4} n^{-\frac{1}{2}} \omega_{[0,1]}^{(h_*^\epsilon)'}(n^{-\frac{1}{2}}) \leq \frac{3}{4} cn^{-1}. \tag{4.22}$$

□

In other words convergence of formula (4.6) is $O(n^{-1})$. This is not extremely fast but probably the best one could hope when dealing with Bernstein polynomials. Indeed, there is strong evidence that

$$|B_n f(x) - f(x)| = o(n^{-1}) \tag{4.23}$$

is false for all non-linear continuous functions f on a real domain (see [12], page 22).

4.2 L^2 Projections

The last pricing methodology we are going to present is the L^2-projections Theorem for continuous payoffs on $[0, +\infty)$ of Carr and Lee, [4].

Similarly to the Bernstein approximation case, we intend to write the payoff $h(x)$ as a limit of an L^2 (*and* uniformly) converging sequence, and then decompose the expectation as a converging sequence of single replicating exponential claims. These claims are nothing but the L^2 projections of h on the span of a suitable subset of a basis of Λ_0. Again, this can be done directly only under the assumption $S_0 \leq K$; otherwise one considers what follows as applied to the claims h^ϵ and has the corresponding approximate pricing.

We denote again by Λ_0 the Banach space of continuous function on $\mathbb{R}^+$ having finite limit. We have the following Proposition:

Proposition 4.6. *Let $h(x)$ in (1.8) be such that $S_0 \leq K$. Let μ be a finite measure on $\mathbb{R}^+$ and P be the $\mathbb{Q}$-distribution of $\langle X \rangle_T$; assume the Radon-Nikodym derivative $dP/d\mu$ is $L^2(\mu)$ and that P is absolutely continuous with respect of μ. Then for all $c > 0$ the solution $\{a_{n,k}\}_{k=1\ldots n}$ of the linear system*

$$\sum_{k=0}^{n} a_{n,k} \langle e^{-cjx}, e^{-ckx} \rangle = \langle h(x), e^{-cjx} \rangle, \quad j = 0, \ldots, n, \tag{4.24}$$

where $\langle , \rangle$ is the inner product in $L^2(\mu)$, satisfies

$$\mathbb{E}[h(\langle X \rangle_T)] = \lim_{n \to +\infty} \sum_{k=0}^{n} a_{n,k} \mathbb{E}[P_k(S_T, S_0)] \tag{4.25}$$

with $P_k(x, y)$ as in Proposition 4.3.

Proof. Let $\mathcal{C} = \{1, x, \ldots, x^n, \ldots\}$ the usual basis for $C([0,1], || \, ||_\infty)$. For $c > 0$ the isomorphism ψ_c^* of (4.4) maps $\mathcal{C}$ into the basis of Λ_0

$$\mathcal{B} = \{1, e^{-cx}, \ldots, e^{-cnx}, \ldots\}. \tag{4.26}$$

Being $h \in \Lambda_0$, for all n let A_n be the L^2 projection of h on the span of $\{1, \dots, e^{-cnx}\}$, that is

$$A_n = \sum_{k=0}^{n} \langle h(x), e^{-ckx} \rangle e^{-ckx} \tag{4.27}$$

and set $a_{n,k} = \langle h(x), e^{-ckx} \rangle$. Being $\mathcal{C}$ dense in $C([0,1], || \, ||_\infty)$ then $\mathcal{B}$ is dense in Λ_0 and so it is in Λ_0 with respect to the $L^2(\mu)$ norm. Hence,

$$\lim_{n \to \infty} A_n(x) = h(x) \tag{4.28}$$

in $(\Lambda_0, || \, ||_2)$. Pick then $\epsilon > 0$ and n_0 such that for all $n > n_0$ it is $||h - A_n||_2 < \epsilon$. Calling $C = \int_{\mathbb{R}^+} (dP/d\mu) d\mu$ we have, by Cauchy-Schwartz inequality

$$\mathbb{E}[h(\langle X \rangle_T) - A_n(\langle X \rangle_T)]^2 = \left[\int_0^\infty \frac{dP}{d\mu}(z)(h(z) - A_n(z)) d\mu(z) \right]^2 \tag{4.29}$$

$$\leq \int_0^\infty \left(\frac{dP}{d\mu}(z) \right)^2 d\mu(z) \quad \int_0^\infty (h(z) - A_n(z))^2 d\mu(z) \leq C\epsilon^2 \mu(\mathbb{R}^+). \tag{4.30}$$

for all $n > n_0$, which shows $\mathbb{E}[A_n(\langle X \rangle_T)] \to \mathbb{E}[h(\langle X \rangle_T)]$. Applying Proposition 1.4 yields (4.25). Moreover, for all $j = 1, \dots, n$ it is

$$\langle h(x), e^{-cjx} \rangle = \langle A_n(x), e^{-cjx} \rangle = \sum_{k=0}^{n} a_{n,k} \langle e^{-cjx}, e^{-ckx} \rangle. \tag{4.31}$$

□

As in the case of the Bernstein polynomials, everything boils down to the calculation of certain constants $a_{n,k}$. An approximated solution to the system (4.24) could be obtained by using least-square approximations on the regressors e^{-cjx} of weight μ. Proposition 4.6 then gives us, under the constraints given in its assumptions, a certain degree of freedom in the choice of such a weight. Again, varying c effects the speed of convergence.

5 Some numerical testing and conclusions

The pricing results have been tested with MATLAB in a typical Heston model scenario for stochastic volatility, using as a benchmark a Monte Carlo simulation of order $n = 10.000$.

$$dS_t = v_t^{1/2} S_t dW_t, \quad S_0 = 100 \tag{5.1}$$

with the underlying CIR process for the variance given by the SDE

$$dv_t = \kappa(\theta - v_t) + \eta v_t^{1/2} S_t dZ_t, \quad v_0 = 0.2, \tag{5.2}$$

W_t and Z_t being independent Brownian motions. The mean reverting κ rate has been set to 0.5, the mean reverting level θ is 0.2 and the volatility of volatility η equals 0.3.

We fix throughout a target volatility level $\overline{\sigma} = 0.1$ and see how the Taylor approximation, the Laplace transform and the uniform convergence perform for various strikes and maturities. We also comment on the sensitivities of the various approximations to the variation of K and T.

Table 1: **An overview of the performance of the different methods, maturity T=1.**

Strike	60	80	100	120	140
Taylor n=4	9.9064	6.4018	3.9878	2.4386	1.4575
Laplace transform	9.7790	6.3622	3.9565	2.4135	1.4719
Bernstein polynomial n=30	10.3676	6.6147	3.9558	2.3117	1.4011
Monte Carlo	9.7550	6.3512	3.9557	2.4132	1.4682

5.1 Taylor polynomials

The analytic expressions of Theorem 2.1 for Taylor polynomials in K up to order 4 are considered. We first analyse the change in value of the TVO as a function of the strike K.

Figure 1: Plot in strike for maturity T=0.25.

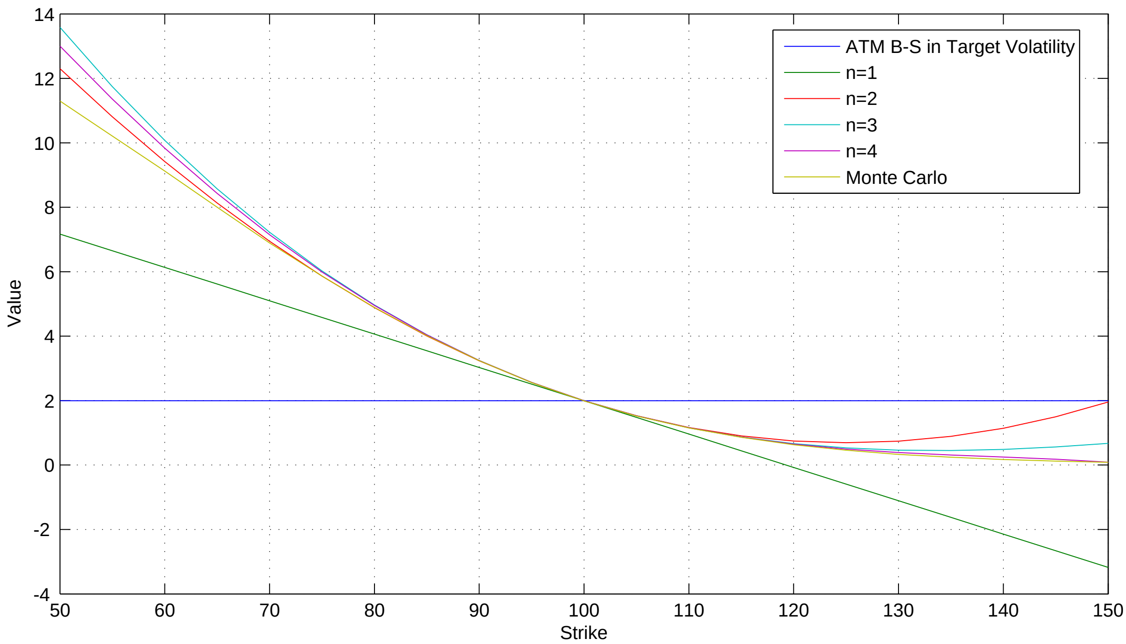


Figure 2: Plot in strike for maturity T=1.

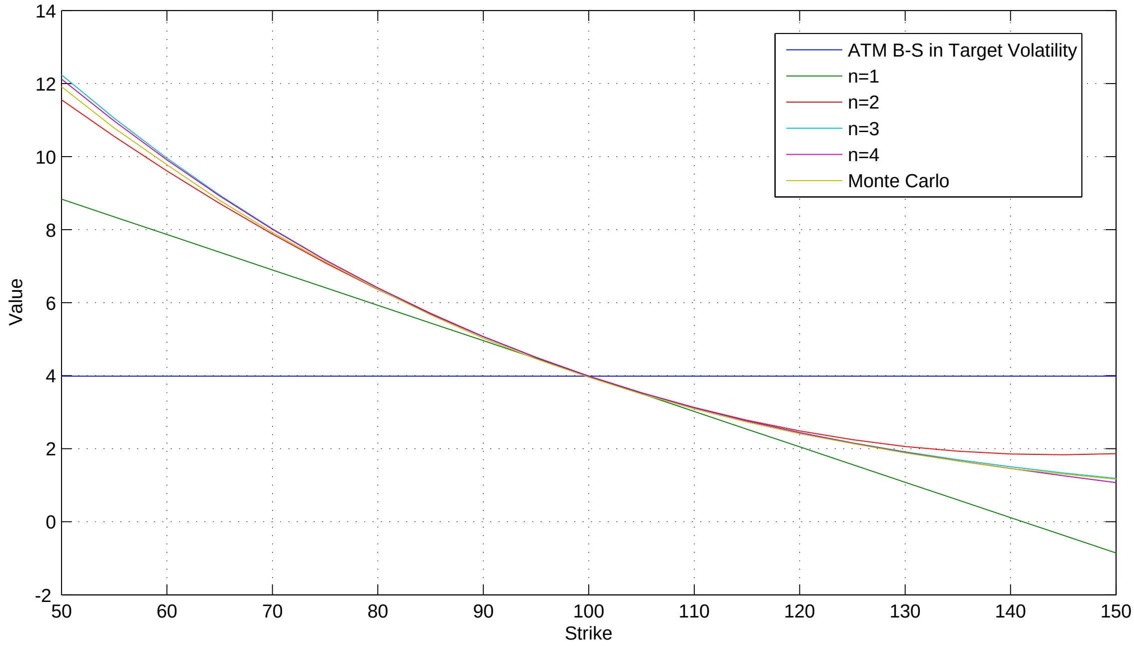
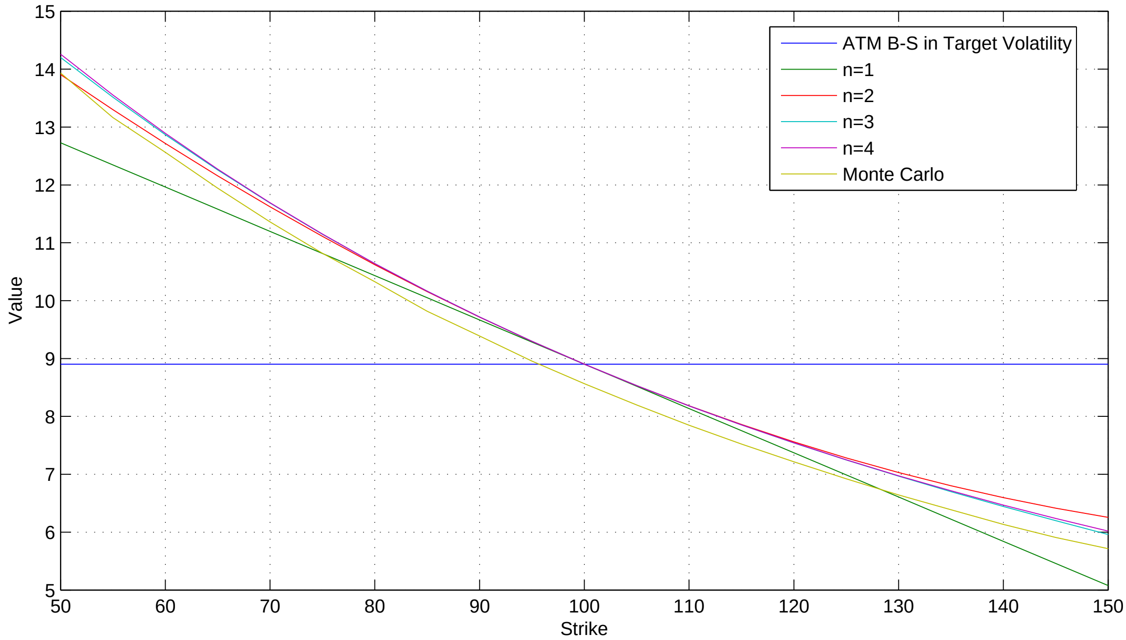


Figure 3: Plot in strike for maturity T=5.



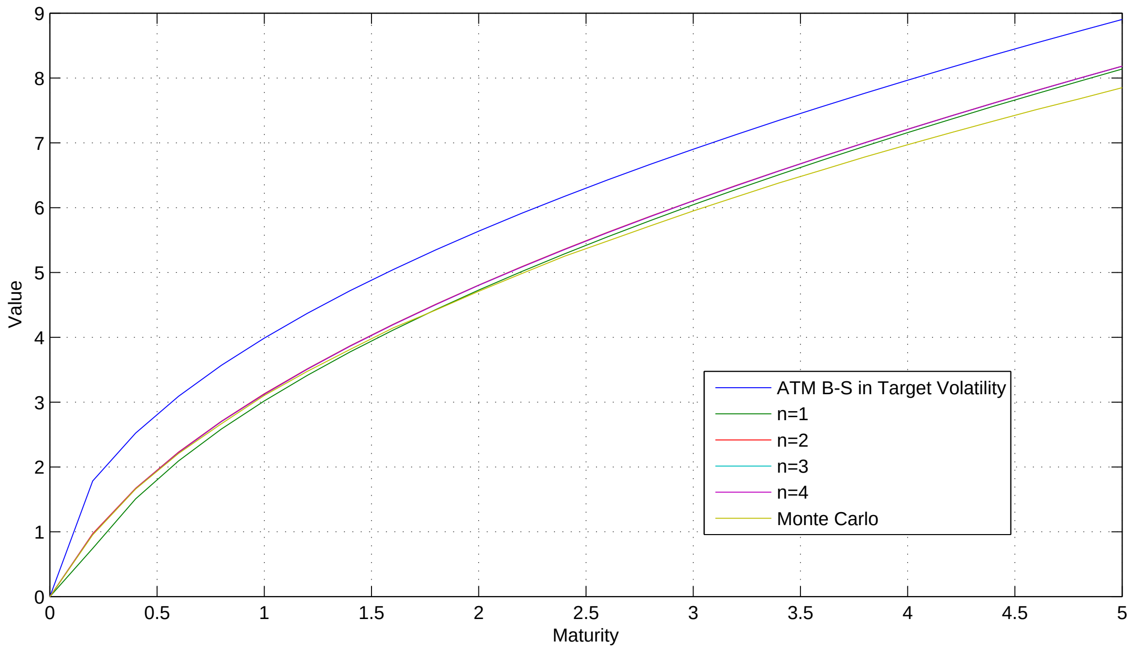
In both cases $T = 0.25$ and $T = 1$ the approximations performs excellently. In figure 1 the approximation is slightly less accurate in the tail; the reason being that for $T \to 0$

the option value approaches its non-analytic intrinsic value and this results in diminished accuracy away from a neighbourhood of S_0.

For $T = 5$ performance is effected by the cumulation with T of a source of error. As time progresses realised volatility constantly increases; having the the at-the-money Black Scholes formula negative second derivative in volatility, a concavity correction starts to show. When simulating a long term process extreme volatility events start to become more likely and this correction effectively reduces the averaged value for $h(x)$. This ultimately results in the Monte Carlo curve of figure 3 being translated backwards respect to the polynomial curves.

The Taylor pricing method shows in fact the asymptotic behaviour in maturity shown in figure 4. Observe how the Monte Carlo simulation bends more rapidly than the Taylor polynomials.

Figure 4: Plot of the value for increasing maturity of an out-of-the-money option, K=110.



Following are some numbers from the graph; it is interesting to notice how the linear approximation already achieves remarkable accuracy.

Table 2: **Values of the Taylor polynomials for T=1 compared to the Monte Carlo simulation.**

Strike	ATM value	n=1	n=2	n=3	n=4	Monte Carlo
90	3.9878	4.9568	5.0656	5.0711	5.0709	5.0550
95	3.9878	4.4723	4.4995	4.5002	4.5002	4.4714
100	3.9878	3.9878	3.9878	3.9878	3.9878	3.9566
105	3.9878	3.5032	3.5305	3.5298	3.5298	3.4985
110	3.9878	3.0187	3.1276	3.1221	3.1219	3.0898

5.2 Laplace transform

The inversion of Corollary 3.4 has been performed with the already mentioned algorithm of Abate-Whitt ([1]), and figures have proven to be extremely precise. We include the graphs in the strike variable for fixed maturities $T = 0.5$ and $T = 3$. A graph for fixed strike and increasing maturity is also included.

Figure 5: Graph of the value against the strike, T=0.5.

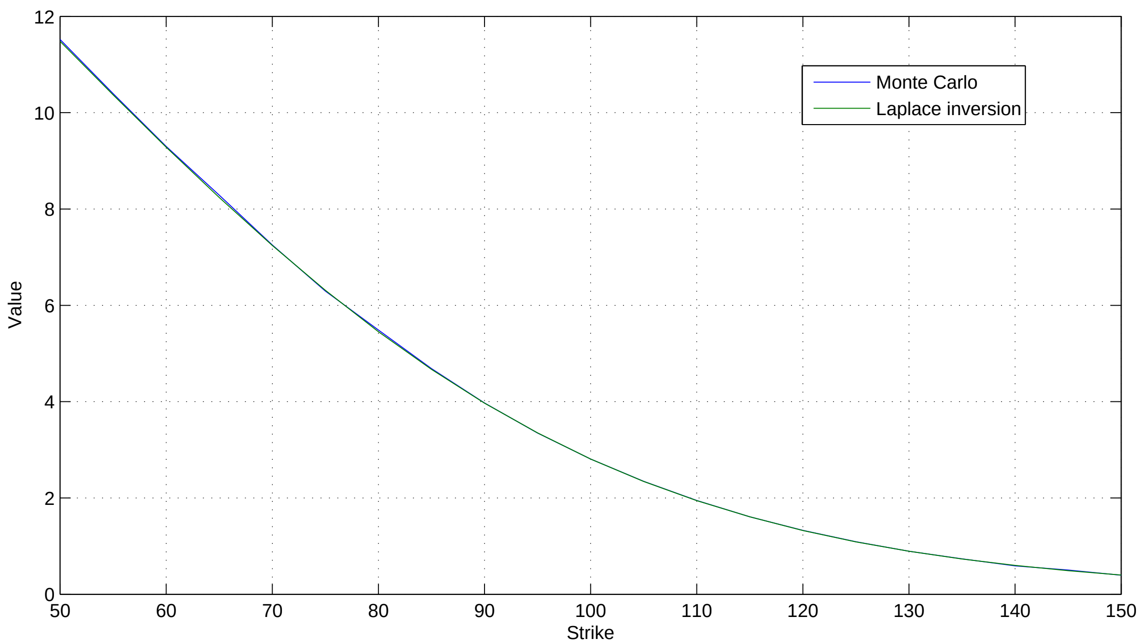


Figure 6: Graph for the value against the strike, T=3.

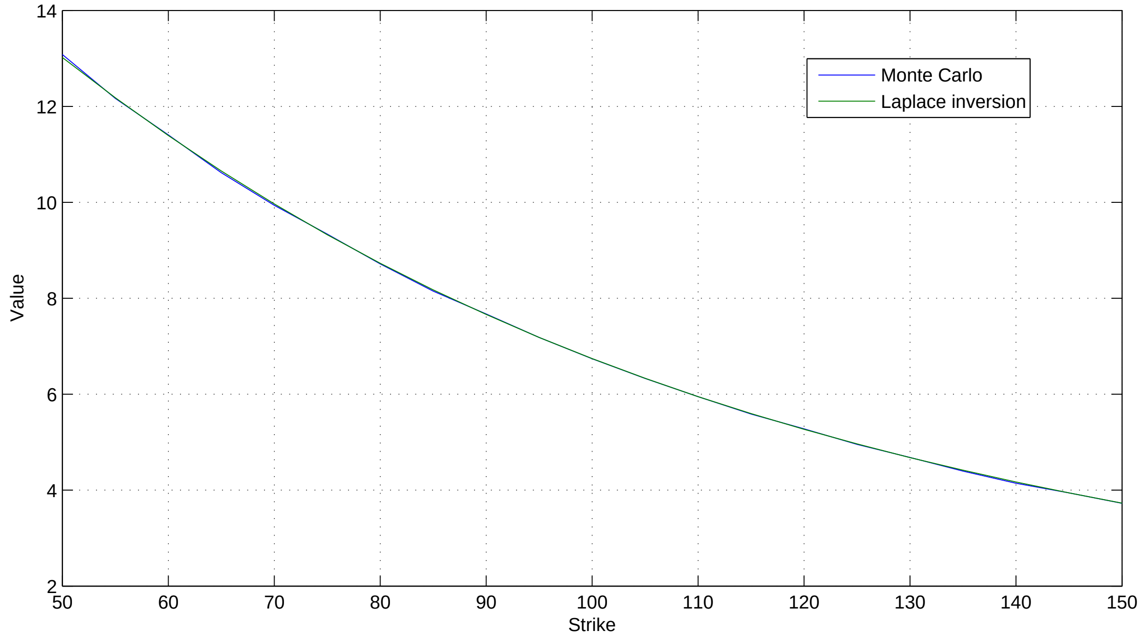


Figure 7: ATM value for increasing maturity.

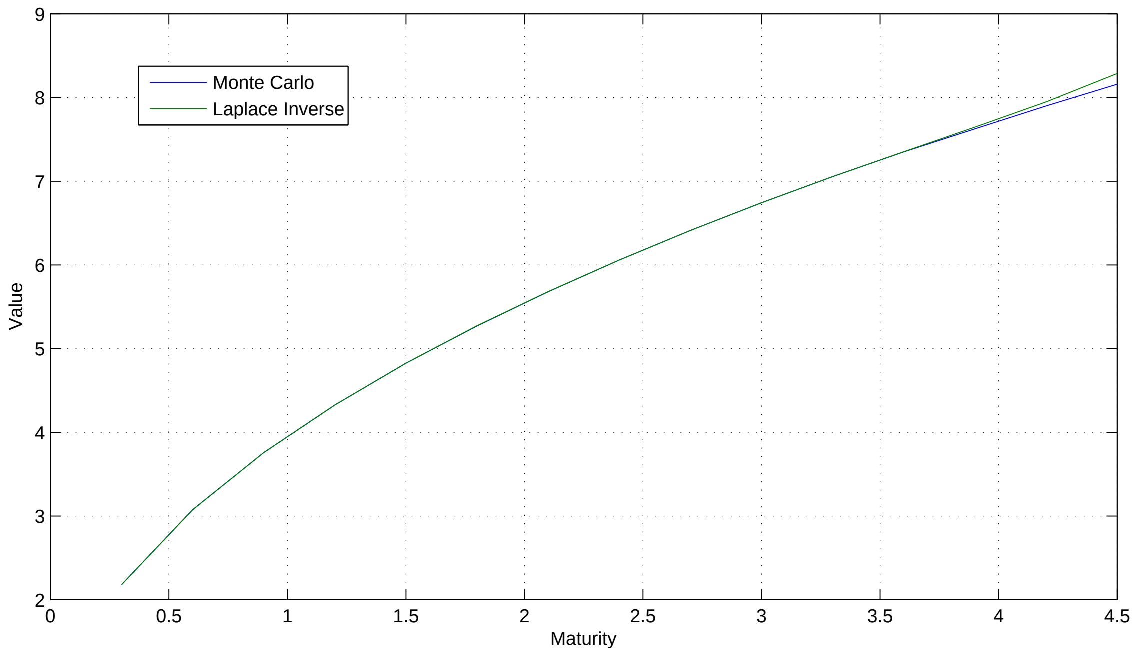


Table 3: **Values of the Laplace inversion method compared to the Monte Carlo simulation, T=3.**

Strike	Inversion	Monte Carlo
90	7.6715	7.6619
95	7.1843	7.1846
100	6.7389	6.7417
105	6.3286	6.3308
110	5.9473	5.9495

The validity of the formulas of subsection 3.2 is therefore confirmed for all strike and maturity ranges.

5.3 Uniform convergence

Following are the figures for the uniform convergence of claims of Proposition 4.3. For short maturities and low strikes the value of the option is rapidly decreasing becoming very steep near 0, thus we need a high degree polynomial to obtain a good approximation, as is the case of figure 9.

Figure 8: Plot for the value against the strike, T=0.5.

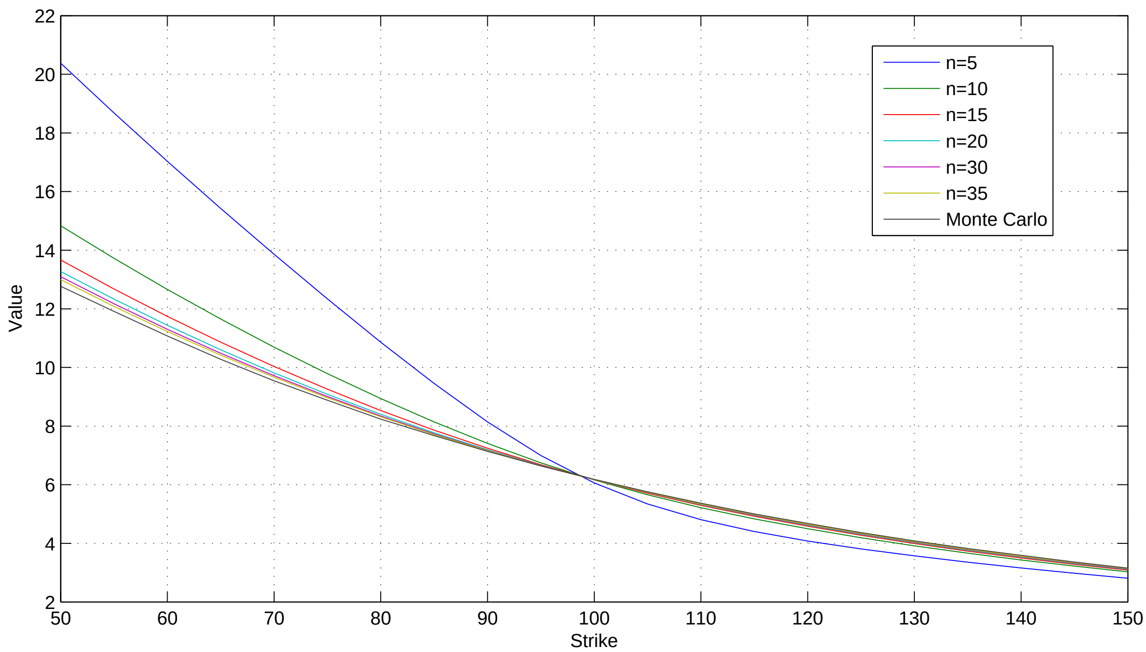


Figure 9: Plot for the value against the strike, T=3.

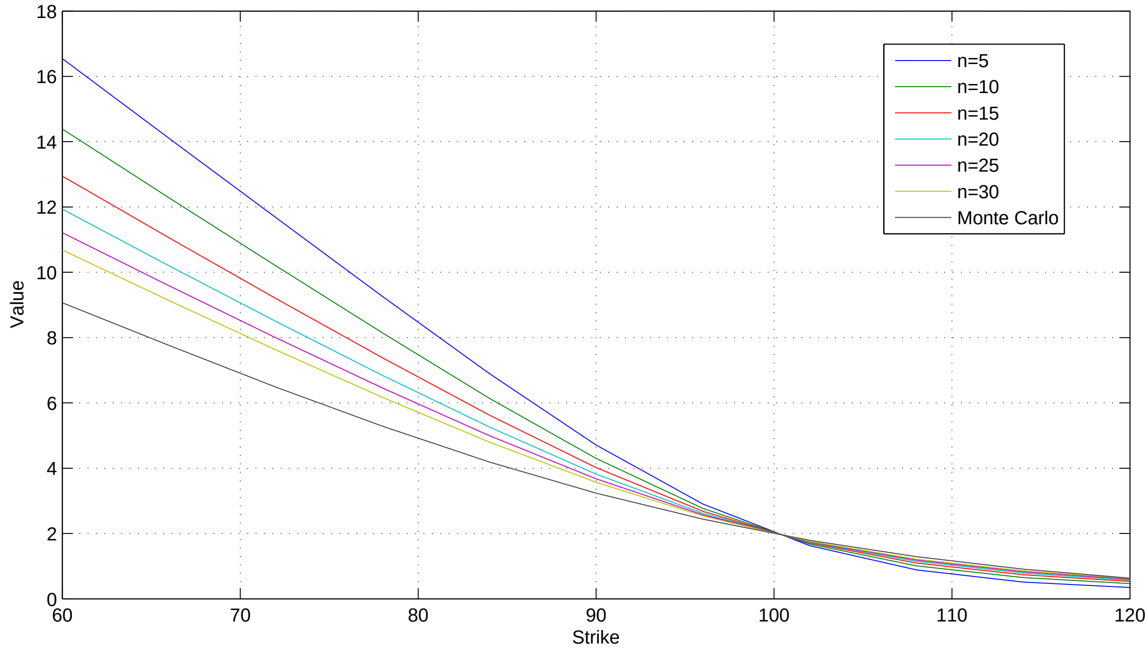
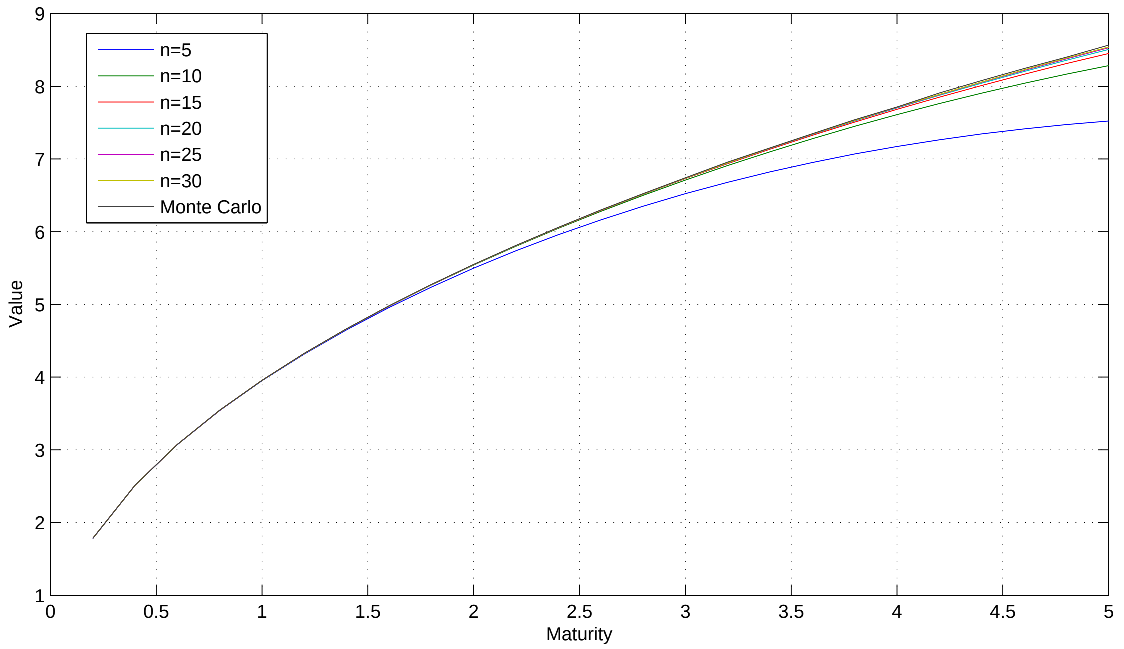


Figure 10: ATM values for increasing maturity.



When maturity increases the price function smoothens out and convergence improves resulting in great accuracy. We cannot expect anyway a polynomial of fixed degree to represent an equally good approximation for different maturities. As figure 10 shows, all polynomials start to decline eventually, while the option value is increasing in T: this is because the characteristic function of $\langle X\rangle_T$ in the CIR model tends to 0 when $T \to +\infty$.

Table 4: **Bernstein polynomials compared to the Monte Carlo simulation, T=2.5.**

Strike	n=5	n=10	n=15	n=20	n=25	n=30	Monte Carlo
90	8.1430	7.4075	7.2448	7.1922	7.1692	7.1567	7.1373
95	6.9970	6.7454	6.6809	6.6594	6.6499	6.6447	6.6395
100	6.0604	6.1618	6.1702	6.1723	6.1733	6.1739	6.1790
105	5.3411	5.6558	5.7101	5.7281	5.7367	5.7416	5.7570
110	4.8056	5.2181	5.2959	5.3234	5.3369	5.3449	5.3658

5.4 Conclusions and future work

Several different solutions have been proposed for the TVO pricing problem, which can be divided into three main categories

- Taylor polynomial approximation;
- Analytical expression for Fourier and Laplace transform and pricing via inversion;
- Approximations via converging sequences: uniform and L^2;

Most of these methods have been closely examined and implemented numerically in a concrete CIR stochastic volatility model and the figures confirm our theory.

Aspects that are currently under inspection and for which it is necessary to gain further insight are:

1. behaviour of our formulas under presence of correlation between the underlying asset and the volatility;
2. choice of a more comprehensive set of assumptions replacing or improving (**A**) and (**B**);
3. further numerical implementation.

With respect of 1 the main reference is still the paper of Carr and Lee, [4]. An analogous of their replicating correlation-neutral claims for the payoff $h(x)$ on the variance can be derived, but it is unclear whether our original payoff (1.4) will still be close in value to a modified correlation-neutral version of (1.8).

The motivation for 2 is to grant the existence of (1.5) under the minimal assumptions on the relevant stochastic processes. For (3) the FFT method of [5] is being implemented for Corollary 3.2 while practical approaches to Proposition 4.6 still need to be clarified.

References

[1] Joseph Abate and Ward Whitt. Numerical inversion of laplace transforms of probability distributions. *ORSA Journal on Computing*, 7:36–43, 1995.

[2] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions.* Dover, 1970.

[3] David Breeden and Robert Litzenberger. Prices of state contingent claims implicit in options prices. *Journal of Business*, 51:621–651, 1978.

[4] Peter Carr and Roger Lee. Robust replication of volatility derivatives. Mathematics in finance working paper series, New York University, Courant Institute of Mathematical Sciences, 2008-3.

[5] Peter Carr, Dilip B. Madan, and Robert H Smith. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 2:61–73, 1999.

[6] Peter Carr, Morgan Stanley, and Dilip Madan. Towards a theory of volatility trading. In *Reprinted in Option Pricing, Interest Rates, and Risk Management, Musiella, Jouini, Cvitanic*, pages 417–427. University Press, 1998.

[7] Daniel Dufresne. The integrated square-root process. Research Paper 90, University of Montreal, 2001.

[8] Arturo Estrella. Taylor, Black and Scholes: series approximations and risk management pitfalls. Research Paper 9501, Federal Reserve Bank of New York, 1995.

[9] Peter Fritz and Jim Gatheral.

[10] Jim Gatheral. *The Volatility Surface: a Practitioner's Guide.* Wiley Finance. Wiley, 2006.

[11] John Hull and Alan White. The pricing of options on assets with stochastic volatility. *Journal of Finance*, 42(2):281–300, June 1987.

[12] G.G. Lorentz. *The Bernstein Polynomials.* AMS, 1986.

[13] Walter Schachermayer and Josef Teichmann. How close are the option pricing formulas of Bachelier and Black-Merton-Scholes? *Mathematical Finance*, 18(1):155–170, January 2008.