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by

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Serre Duality for Complex Analytic Manifolds and Projective n-Spaces

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A Gabriella
e Iolanda

Introduction

Due punti determinano una retta
due rette determinano un punto.

This thesis has been conceived to be as broad as possible.

When I first started looking for a subject for my master thesis the thing I had clear in my mind is that I wanted to dissert on something which could have been a *summa* of my studies during the five past years; a work meant as an occasion to gather ideas and elements from various areas of mathematics and let them interact in a whole context, remaining understood that I am taking a major in geometry.

The Serre Duality Theorem gave me the opportunity to do so. A classic theorem in algebraic geometry, maybe the most important tool in cohomology sequences calculations, yet extremely subtle and meaningful in its own sake if one wants to prove it in greater generality: as we shall see throughout the dual vector space of $H^n(X, \mathcal{F})$ has a different nature depending on who are X and $\mathcal{F}$. If we are working with locally free sheaves in an algebraic category, or, in view of the equivalence between algebraic curves and compact Riemann surface, X is analytic of dimension 1, the Theorem is not so difficult. An easy proof can be found for example in [18].

If one works with analytic manifolds or rises the dimension the situations starts looking more complicated: this difference can be clearly seen by looking at Serre's work, which presented the theorem in two distinct papers, [23] and [24], issued in the years 1954-1955, one for the case of algebraic projective varieties, and the other one for analytic manifolds.

The reason of such a distinction lies in the fact that when considering manifolds the algebras of holomorphic functions are no longer finitely generated algebras (as they are in the case of varieties), so that unlikely cohomologies of locally free sheaves (and their duals) are finitely-dimensional vector spaces: that is where functional analysis comes in. For the reasons and purposes I explained I choosed to put the emphasis on the analytic case: indeed the proof of the Duality Theorem in this setting involves complex calculus, topological vector space theory, functional analysis and complex analytic geometry. Moreover the analytic proof is in my opinion more revealing of what is going on, because by going through it one can, fastening his intuition to differential forms, effectively realize that it has been worked out by a duality relation already existing between two complexes, both resolving the sheaf of holomorphic p-forms.

There are however two main downsides to this approach. The first is that the Duality Theorem cannot be valid for arbitrary manifolds: this claim may cause disbelief in those who are accustomed to the algebraic case, but I hope I will be enough convincing in section 2.3.

The second is that one should then only take into account locally free sheaves, the reason being that tensoring with a locally free sheaf locally does not alterate the topological structure of the space of the differential forms, whose topology is the stepping stone of the duality. Actually I do not believe this proof can be extended to coherent sheaves but because in that case derived functors have to be dragged in.

To remedy to this certainly unsatisfying limitation I added a third chapter where the complete proof of the algebraic case is carried over: this allowed me to dissert a little on some foundations of scheme theory and elements of homological algebra, to remain in the interdisciplinary spirit of this dissertation. Even tough completely explained, this part remains secondary in the general framework, being the algebraic proof mostly crafted by means of mysterious, yet fascinating, homological facts, therefore less suitable for a graduation thesis.

The spirit in developing the background elements has been the following. What I have assumed without explanation are facts from commutative algebra, sheaf theory and vector bundles, elements of category theory, complex analysis of one variable and basic cohomology theory, as well as some popular functional analysis theorems as the Hahn-Banach Theorem, and whatever else I felt I already knew fairly good. On the other hand I tried to include in the discussion everything that was rather new to my knowledge; this is the reason why within these pages there is far more than what needed to understand the Theorem. Admittingly some pages are not strictly connected to the Serre Duality but I think that from a didactical point of view it has been positive to include as much as possible, while trying to stay focused on the subject.

The first chapter is a preliminary dicussion on the concepts needed to understand the Theorem's analytic statement. The main object of interest here is the sheaf of complex differential (p, q) forms, which is thoroughly discussed *ab initio*, starting from the multilinear algebra. Other foundational material introduced are the basic theory of topological vector spaces, currents as dual elements of the differentials, properties of fine resolutions, harmonic forms. Also, the third chapter begins with a concise but I hope complete discussion on schemes and their basic properties, sheaves of modules, projective spaces and derived functors.

The second chapter deals directly with the proof of the Duality for which I respectfully followed [24]. The choice of following the original pattern, while giving to this dissertation a nice philological flavour, in my opinion also leads to a deeper understanding of the subject. The first part of the chapter illustrates how the sheaves of differential (p, q) forms and (p, q)

currents (sequences 2.1 and 2.2) resolve the sheaf of holomorphic p-forms. This was a gap to be filled, since in the article Serre just refers to Schwartz's work ([21]) to support his claim and does not provide proofs. The exactness for forms is quite straightforward and is proved using the Poincaré Lemma for complex (p, q) forms (the proof is that of [13]). I worked out on my own the (much more complicated) exactness of the currents sequence, by combining the hints given in chapter X of [10] with the complex calculus techniques of [16].

At the end of the chapter some remarkable applications are discussed: the Riemann Roch Theorem, the compact case, and the classical applications to divisors on a Riemann Surfaces. A counterexample is provided among the class of those domains in $\mathbb{C}^n$ that are not domains of holomorphy (that is, manifolds embedded in $\mathbb{C}^n$ that are not Stein manifolds).

The third chapter begins with the mentioned review on schemes and homological algebra, proves the Theorem, as in [14], for the projective n-space, meant as a scheme over an algebraically closed field k (even if the same proof should work for $\mathbb{P}^n_A$). A few words are also dedicated to the concept of dualizing sheaf.

I would like to thank everyone who helped me during my efforts, most notably professor Edoardo Sernesi, professor Ugo Bessi, Lorenzo Di Biagio, Dajano Tossici and Carmelo Finocchiaro.

Contents

Chapter 1

Introductory Material

This chapter aims at developing the basic tools needed to understand the Serre Duality; we start from multilinear algebra in order to introduce differential and (p, q) forms in section 1.2. Elements of topological vector spaces theory are in section 1.3. In 1.4 we deal with currents and distributions; analytic manipulations of these objects (smoothing) are explained together with the definition of a Laplace operator for forms (section 1.5). The fact that both forms and currents on a manifold are fine sheaves plays a major role in the proof of the Duality, we will also provide some general cohomological results on fine sheaves.

1.1 Some Multilinear Algebra

All vector spaces are finite dimensional over a field K. all algebras commutative. Let V be a vector space; we set

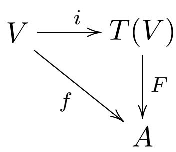
$$T^k(V) = \underbrace{V \otimes V \ldots \otimes V}_{k \text{ times}}$$

we call an element of $T^k(V)$ a *tensor* in V and set $T^0(V) = K$ and $T^1(V) = V$ by definition. Let

$$T(V) = \bigoplus_k T^k(V);$$

this is an algebra over V, with multiplication given by tensor product, that we call the *tensor algebra* of V. $T(V)$ satisfies the following universal property.

Proposition 1.1.1. *Let V be a vector space and A a K-algebra. The pair $(i, T(V))$, with i the identification of V with $T^1(V)$, is such that for each K-linear map $f : V \to A$ there is a* *unique algebra morphism $F : T(V) \to A$ such that the diagram*



commutes.

Proof. For each k let V^k be the cartesian product $\overbrace{V \times \ldots \times V}^{k \text{ times}}$ and $F_k : V^k \to A$ defined by

$$F_k(v_1, \ldots, v_k) = f(v_1) \ldots f(v_k).$$

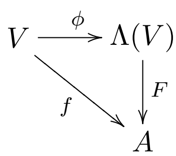
In particular $F_1 = f$ and F_0 is the inclusion of the field $V^0 = K$ in A. By the universal property of the tensor product, F_k extends to a unique linear mapping $\overline{F_k} : T^k(V) \to A$. Now set $F := \sum_k \overline{F}_k$; we have for each $t, s \in T(V)$, that $t = \sum_{k \geq 0} t_k$, $s = \sum_{k \geq 0} s_k$, for certain $t_k, s_k \in T^k(V)$ and

$$\begin{aligned} F(t \otimes s) &= F\left(\sum_k t_k \otimes \sum_h s_h\right) = \sum_j \overline{F}_j \left(\sum_k t_k \otimes s_{j-k}\right) = \\ &= \sum_j \sum_k \overline{F}_j(t_k \otimes s_{j-k}) = \sum_j \sum_k \overline{F}_k(t_k)\overline{F}_{j-k}(s_{j-k}) = \\ &= F(t)F(s) \end{aligned}$$

is an algebra morphism and we see that if $v \in V$ then $f(v) = F_1(i(v)) = F(i(v))$. Uniqueness follows from uniqueness of $\overline{F}_k$ and from the graded structure of $T(V)$. □

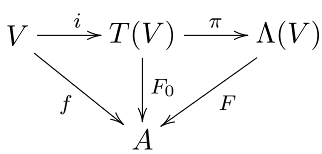
The quotient $\Lambda(V) = T(V)/I$ with I the ideal generated in $T(V)$ by all the elements in the form $v \otimes v$, is the *exterior algebra* of V. $\Lambda(V)$ is again described universally.

Proposition 1.1.2. *Let V be a vector space, A a K-algebra. There exists a pair $(\phi, \Lambda(V))$, with ϕ a morphism of algebras $\phi : V \to \Lambda(V)$ such that for each linear map $f : V \to A$ such that $f(v)^2 = 0$ there exists a unique algebra morphism F such that the diagram*



commutes.

Proof. Let $\Lambda(V)$ be the quotient above and consider the diagram



where π is the projection and F_0 as in the previous proposition; if $F_0(I) = 0$ then F_0 descends uniquely to algebra morphism F on the quotient and the result follows by letting $\phi = \pi \circ i$. Indeed if $v = v_1 \otimes \ldots \otimes v_k$ is any tensor in $T^k(V)$ being F_0 an algebras morphism and satisfying the universal property

$$F_0((v_1 \otimes \ldots \otimes v_k) \otimes (v_1 \otimes \ldots \otimes v_k)) = F_0(v_1 \otimes \ldots \otimes v_k)^2 = [f(v_1)^2 \ldots f(v_k)^2] = 0$$

□

The induced product $v \wedge w := \pi(v \otimes w)$ on $\Lambda(V)$ is called the *wedge* product. It inherits the associativity and the homogeneity properties of the tensor product; furthermore it is skew-symmetric. For $v, w \in \Lambda(V)$

$$0 = (v + w) \wedge (v + w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w \Rightarrow v \wedge w = -w \wedge v.$$

Take a basis $(e_1, \ldots, e_n)$ of V and consider in $\Lambda(V)$ all the subspaces in the form $\Lambda^k(V) = \text{Span}\{e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{i_1,\ldots,i_k}$ for $i_1, \ldots, i_k \in \{1, \ldots, n\}$ and we set $\Lambda^0(v) = K$, $\Lambda^1(V) = V$. Being $e_{i_k} \wedge e_{i_h} = 0$ whenever $i_h = i_k$ by skew-symmetry, $\Lambda^k(V)$ is spanned by wedge products of basis elements indexed in a strictly increasing way $1 \leq i_1 < \ldots < i_k \leq n$, so that $\dim\Lambda^k = \binom{n}{k}$. Moreover $\Lambda^k(V) = 0$ if $k > n$ and any two of these subspaces are disjoint, so we have a decomposition

$$\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V).$$

$\Lambda^k(V)$s are called *k-th exterior power of V*. We have that if $v \in \Lambda^k(V)$ and $w \in \Lambda^h(W)$, $v \wedge w \in \Lambda^{k+h}(V)$ with $v \wedge w = (-1)^{kh} w \wedge v$.

We can give a description of these vector spaces in another way. A k-tensor $v_1 \otimes \ldots \otimes v_k$ is said to be *alternating* if for any permutation of k elements σ

$$v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)} = sgn(\sigma) v_1 \otimes \ldots \otimes v_k.$$

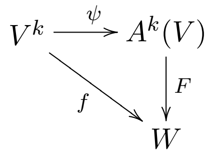
The set of all such elements is a vector subspace of $T^k(V)$ that we denote $A^k(V)$. The linear mapping $Alt_k : T^k(V) \to A^k(V)$ acting by

$$Alt(v_1 \otimes \ldots \otimes v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(k)}$$

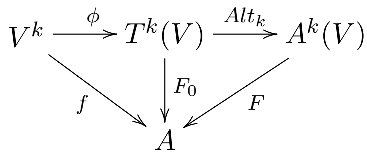
Clearly if $v_1 \otimes \ldots \otimes v_k \in A^k(V)$ we have $Alt_k(v_1 \otimes \ldots \otimes v_k) = v_1 \otimes \ldots \otimes v_k$ so that in particular the 'alternator' Alt_k is surjective.

Recall that a linear map f between vector spaces is said to be *alternating* if whenever $v_1, \ldots, v_k$ are linearly dependent $f(v_1, \ldots, v_k) = 0$. $A^k(V)$ satisfies the following universal property:

Proposition 1.1.3. *Let V, W be vector spaces. There is an alternating map $\psi : V^k \to A^k(V)$ such that for any alternating linear map $f : V^k \to W$ there exists a unique linear map $F : A^k(V) \to W$ that fills the diagram*



Proof. As before by the universal property of the tensor product exists a unique linear map F_0 which factors the diagram into



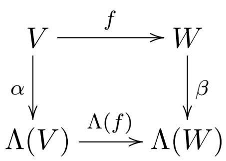
If such a filler F exists then it is unique because it must be $F_0 = F \circ Alt_k$ and F_0 is unique. Set $\psi = Alt_k \circ \phi$ where ϕ is the homomorphism from the cartesian product to the tensor product; it is alternating because Alt_k is. Define for $v \in A^k(V) \subset T^k(V)$, $F(v) = F_0(v)$; then for $v = (v_1, \ldots, v_n)$

$$f(v) = F_0 \circ \phi(v) = F \circ Alt_k \circ \phi(v)$$

if and only if $F_0(w) = Alt_k \circ F(w)$ for $w \in A^k(V)$, that is, if and only if $F_0 = F_0 \circ Alt_k$. But this follows, as already noticed, from the fact that $Alt(A^k(V)) = A^k(V)$. □

This links to the preceding definition. The exterior powers $\Lambda^k(V)$ can be written as a quotient $T^k(V)/I^k(V)$ where $I^k(V) = \{v_1 \otimes \ldots \otimes v_n | \, v_i = v_j, \text{ exist } i, j\}$; a rather tedious computation shows that $I^k(V) = ker(Alt_k)$ yielding to $A^k(V) \cong \Lambda^k(V)$.

This construction has been carried out to outline the *functoriality* of $\Lambda(V)$ and $\Lambda^k(V)$. Indeed given a linear map $f : V \to W$ of vector spaces call $\alpha : V \to \Lambda(V)$ and $\beta : W \to \Lambda(W)$ the canonical inclusions of V and W, viewed as Λ^1. By proposition 1.1.2 there exist a unique morphism of algebras $\Lambda(f) : \Lambda(V) \to \Lambda(W)$ that fills $f \circ \beta$, that is such that the diagram



commutes. The other functorial properties are easily checked. Of course by applying proposition 1.1.3 the same holds for Λ^k. So we have two functors $\Lambda, \Lambda^k(V) : Vect_K \to Alg_K$ from the category of finite dimensional vector spaces on K to that of finite rank algebras on K. In our context this comes into play while dealing with vector bundles

In general given a (differentiable or holomorphic) vector bundle $\pi : E \to M$ with fibre E_p and transition functions $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(n, K)$ subordinated to a covering $\{U_\alpha\}$, and a functor $F : Vect_K \to Vect_K$, we can canonically define a bundle $\pi : F(E) \to M$ whose fibres are $F(E_p)$ and transition functions on $\{U_\alpha\}$ given by $F(\phi_{\alpha\beta})(p) = F(\phi_{\alpha\beta}(p))$.

So starting from a given bundle $\pi : E \to M$ on a manifold M of dimension n, we may define an *exterior bundle* $\Lambda(E)$ of rank 2^n, or a *k-th exterior bundle* $\Lambda^k(E)$ of rank $\binom{n}{k}$. Other examples of bundles obtained by functors are the direct sum bundle $E \oplus F$ or the tensor bundle $E \otimes F$.

We can already define the geometric invariant that plays a fundamental role in the Serre Duality. We recall that a *holomorphic (differentiable) section* of a bundle $\pi : E \to M$ on an open set $U \subset M$ is a holomorphic (differential) map $s : U \to E$ such that $\pi \circ s = id_U$.

Definition 1.1.4. Let T^*X be the holomorphic cotangent bundle of some n dimensional complex manifold M. The bundle $\Lambda^n(T^*X)$ is a rank 1 vector bundle called the *canonical* or *determinant* bundle of M. The sheaf of *holomorphic* sections associated with this bundle is the *canonical sheaf* $\mathcal{K}$ of M.

It is also worth noting that by the universal property of the tensor product we have an isomorphism from the vector space $\mathcal{L}(V^k, K)$ of the k-multilinear forms on from the vector space V to K with $T^k(V^*)$. So really a k-tensor on V^* is just a tensor in the geometric meaning (a multilinear functional) and in this fashion $\Lambda^k(V^*)$ turns out to be just the algebra of alternating k-tensor that is, the subspace of those tensors $T : V^k \to K$ such that for each $v_1, \ldots, v_k$ and for each $\sigma \in S_k$

$$T(v_1, \ldots, v_k) = sgn(\sigma) T\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right).$$

We now give a result that will be useful later

Proposition 1.1.5. *Let V, W be two vector spaces. Then*

$$\Lambda^k(V \bigoplus W) = \bigoplus_{p+q=k} \Lambda^p(V) \otimes \Lambda^q(W)$$

Proof. We proceed by induction on k. If $k = 1$ then

$$\Lambda^1(V \oplus W) = V \oplus W = (\Lambda^1(V) \otimes K) \oplus (K \otimes \Lambda^1(W))$$

and the statement follows from the fact that $\Lambda^0 = K$. Since every element $\omega \in \Lambda^{r+s}(v)$ admits a decomposition $\omega = \eta \wedge \gamma$ with $\eta, \gamma \in \Lambda^{r,s}(V)$ we can write $\Lambda^{r+s}(V) = \Lambda^r(V) \wedge \Lambda^s(V)$ by meaning all the possible wedge products of the two spaces; so using induction and distributing the wedge product over the sum

$$\Lambda^k(V \oplus W) = \Lambda^{k-1}(V \oplus W) \wedge (V \oplus W) = \bigoplus_{p+q=k-1} \Lambda^p(V) \otimes \Lambda^q(W) \wedge (V \oplus W).$$

Distributing the sum over the tensor product and arranging correctly the wedges we then have

$$\bigoplus_{p+q=k-1} \Lambda^p(V) \otimes \Lambda^q(W) \wedge (V \oplus W) = \bigoplus_{p+q=k-1} \Lambda^{p+1}(V) \otimes \Lambda^q W \oplus \bigoplus_{p+q=k-1} \Lambda^p(V) \otimes \Lambda^{q+1}(W).$$

By merging into one sum and changing the index we finish the proof. □

1.2 Differential Forms and (p, q) Forms

Differential forms naturally arise as differentiable tensor fields, regular maps from a manifold that associate with a point a tensor from the tangent space at that point, in a coordinate-compatible way.

A differentiable (resp. analytic) complex manifold M of dimension n is, here and ever, a second-countable topological Hausdorff space, locally diffeomorphic (resp. biholomorphic) to an open subset of $\mathbb{C}^n$.

Definition 1.2.1. Let M be a differentiable manifold and T^*M its cotangent bundle. For each k we call the bundle $\Lambda^k(M) := \Lambda^k(T^*M)$ the *k-exterior bundle* of M. A *k-differential form* on an open set $U \subset M$ is a differentiable section of M on U, that is, a differentiable map $s : U \to \Lambda^k(M)$ such that $\pi \circ s = id$.

Note that a 0-form is just a differentiable function and a 1-form a differentiable covector field. Since by definition differential forms are local setions of a differentiable bundle they define a locally free sheaf of modules over the sheaf of rings C^∞_M with rank equal to that of $\Lambda^k(M)$ $\big($ which is $\binom{n}{k}\big)$. This will be denoted $\mathcal{E}^k_M$; we will commonly drop the subscript M.

Recall that a *local frame* over U for a vector bundle E is a set of n sections $s_1, \dots, s_n$ such that the set $\{s_1(p), \dots s_n(p)\}$ spans the fibre E_p for each $p \in U$. We can associate with a vector bundle a collection of frames over a trivialization $\{U_\alpha\}$ of M. In fact if $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$, trivializes E then $s_i^\alpha : U_\alpha \to E$ defined by $p \mapsto \phi_\alpha^{-1}(p, e_i)$, for e_i a coordinate vector, is a frame on U_α. Indeed since $\pi_1 \circ \phi_\alpha = \pi$, with π_1 the projection on U_α, then

$$\pi \circ s_i^\alpha(p) = \pi \circ \phi_\alpha^{-1}(p, e_i) = \pi_1(p, e_i) = p$$

so that s_i^α are sections generating E_p at each fixed p.

Conversely given a smooth collection of frames over a covering of M there is a unique vector bundle associated to this frame; this explains the differentiable locally free structure of the sheaf of sections of a bundle.

Let $x_1, \ldots, x_n$ be coordinates over an open set U on which $\Lambda^k(M)$ trivializes. At each point $p \in U$ the cotangent space has basis $(dx_1|_p, \ldots, dx_n|_p)$ where the dx_is are the differentials of the relative coordinates. In particular we have that $dx_1, \ldots, dx_n$ form a (local) frame for T^*M over U. So locally on U $\{dx_{i_i} \wedge \ldots \wedge dx_{i_n}\}_{1 \le i_1 < \ldots < i_n \le n}$ is a frame for $\Lambda^k(M)$ for which every differential form can be represented on U by an expression

$$\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} f_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

with $f_{i_1 \ldots i_k} \in C^\infty(U)$. We now define an external operator between sheaves $d : \mathcal{E}^k \to \mathcal{E}^{k+1}$ which generalizes the differential of a function (also called differential). We use the following notation: a multi-index I is an ordered k-uple $i_1 \ldots i_k$ with $1 \le i_1 < \ldots < i_k \le n$ and for such an index we set $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. Take a local expression of ω on a coordinate open U:

- If $k = 0$ for $f \in C^\infty(U) = \Omega^0(U)$ take

$$df = \sum_{j=0}^{n} \frac{\partial f}{\partial x_j} dx_j$$

  that is, just the differential of f;

- If $k > 0$ for $\omega = \sum_I f_I dx_I$ set

$$d\omega = \sum_I \sum_{j=0}^{n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I = \sum_I df_I \wedge dx_I.$$

This definition can be extended for an arbitrary open set $V \subset M$ using the sheaf axiom; just take a coordinate covering of V and glue all the images under d of the local expressions of $d\omega$ for that covering. One can easily check that d is uniquely determined by the following properties

1. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for every $\omega \in \mathcal{E}^k$, $\eta \in \mathcal{E}^h$;

2. d commutes with restrictions;

3. $d^2 = 0$.

By these we obtain a cochain complex of sheaves on M

$$0 \to C^\infty \to \mathcal{E}^1 \to \ldots \to \mathcal{E}^n \to 0. \tag{1.1}$$

whose homomorphisms on sections are given by d. If we take global sections and call $d_q : \mathcal{E}^q(M) \to \mathcal{E}^{q+1}(M)$ we define the *de Rham cohomology* of M to be the quotients of d_q-cocycles for d_q-coboundaries that is $H^q_{dR} = \dfrac{ker d_q}{d_{q-1}(\mathcal{E}^{q-1})}$.

Suppose now M to be an n-dimensional complex analytic manifold. If $z_1, \ldots, z_n$ are complex holomorphic coordinates on an open set U we can write $z_i = x_i + y_i$, for $i = 1, \ldots, n$, and take $x_1, \ldots, x_n, y_1, \ldots, y_n$ smooth coordinates of the underlying $2n$-dimensional real differentiable manifold. Consider the complexified tangent and cotangent planes T_pM_c and $T^*_pM_c$ of M at each point $p \in U$; these are nothing the space of all the complex linear combinations of the respective bases $\frac{\partial}{\partial x_1}|_p, \ldots \frac{\partial}{\partial x_n}|_p, \frac{\partial}{\partial y_1}|_p, \ldots, \frac{\partial}{\partial y_n}|_p$ and $dx_1|_p, \ldots dx_n|_p, dy_1|_p, \ldots, dy_n|_p$. When p ranges over U we obtain local frames $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$ and $dx_1, \ldots, dx_n, dy_1, \ldots, dy_n$ for the complexified tangent and cotangent bundles TM_c and T^*M_c.

Now since $x_i = \frac{1}{2}(z_i + \overline{z}_i)$ and $y_i = \frac{1}{2i}(z_i - \overline{z}_i)$, using the chain rule the partial derivatives of a smooth function f in the coordinates $z_i, \overline{z}_i$ express as

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - i\frac{\partial f}{\partial y_j}\right) \tag{1.2}$$

$$\frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right) \tag{1.3}$$

Hence the operators $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n}$ at each point are independent vectors on T_pM_c thus forming a local frame for the tangent bundle; similarly $dz_1, \ldots dz_n, d\overline{z}_1, \ldots, d\overline{z}_n$ that turns out to be a new local frame for the cotangent bundle.

We are interested in T^*M_c; because of the complex analytic structure the transition functions on intersections do not send the differentials dz_i to others in the form $d\overline{w}_i$ for another coordinate system $w_1, \ldots, w_n$ on V, $V \cap U \neq \emptyset$. So we have a decomposition

$$T^*M_c = T^*M_c^{1,0} \oplus T^*M_c^{0,1}$$

where $T^*M_c^{1,0}$ and $T^*M_c^{0,1}$ are bundles associated respectively with the frames $\{dz_1, \ldots, dz_n\}$ and $\{d\overline{z}_1, \ldots, d\overline{z}_n\}$ when the coordinates z_i runs over each possible open set of a coordinate covering of M.

In particular this decomposition holds vectorially on fibres so by applying 1.1.5 and switching to vector bundles, using the functorial properties of $\Lambda^k, \otimes$ and $\oplus$ we have

$$\Lambda^k(T^*M_c) = \bigoplus_{p+q=k} \Lambda^p(T^*M_c^{1,0}) \otimes \Lambda^q(T^*M_c^{0,1}).$$

We call $\Lambda^{p,q}(M)$ the bundle $\Lambda^p(T^*M_c^{1,0}) \otimes \Lambda^q(T^*M_c^{0,1})$. Its sheaf of differentiable sections will be called the *sheaf of forms of type* (p,q) and denoted $\mathcal{E}^{p,q}_M$. The decomposition above carries out a decomposition

$$\mathcal{E}^k_{c,M} = \bigoplus_{p+q=k} \mathcal{E}^{p,q}_M$$

of sheaves, where $\mathcal{E}^k_c$ is the sheaf of sections of the complexified cotangent bundles, that is the sheaf of differential forms on M of total degree k whose coefficients are allowed to take complex values.

If we extend the differential d on $\mathcal{E}^k_c$ by complex linearity we obtain a cochain complex entirely analogue to 1.1, but this does not reflect the complex structure. Nevertheless if we restrict d to a single summand of $\mathcal{E}^k_c$ and if $\pi_{p,q} : \mathcal{E}^k_c \to \mathcal{E}^{p,q}$ is the projection on the summand (p, q) we may define

$$\partial = \pi_{p+1,q} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q} \quad , \quad \overline{\partial} = \pi_{p,q+1} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$$

and furthermore we have $d = \partial + \overline{\partial}$. In fact using multi-indices I and J of length p and q a differential form of type (p, q) is written locally like $\omega = \sum_{I,J} f_{I,J} dz_I \wedge d\overline{z}_J$ so

$$d\omega = \sum_{I,J} \left(\sum_{h=0}^{n} \frac{\partial f_{I,J}}{\partial z_h} dz_I \wedge d\overline{z}_J \wedge dz_h + \sum_{h=0}^{n} \frac{\partial f_{I,J}}{\partial \overline{z}_h} dz_I \wedge d\overline{z}_J \wedge d\overline{z}_h \right)$$

and this sum is unique. Consequently

$$0 = d^2 = \partial^2 + (\partial\overline{\partial} + \overline{\partial}\partial) + \overline{\partial}^2;$$

since each of these three operator maps to a different summand of the decomposition of $\mathcal{E}^{k+2}_c$ they all must be 0. For any fixed p the cochain complex of sheaves

$$0 \to \mathcal{E}^{p,0} \to \mathcal{E}^{p,1} \ldots \to \mathcal{E}^{p,n} \to 0. \tag{1.4}$$

with morphisms induced by $\overline{\partial}$, is called the *Dolbeault* complex. In this sequence we can restrict ourselves to global sections to obtain a new complex whose cohomology is the *Dolbeault cohomology*; in symbols $H^{p,q}_{\overline{\partial}}(M) = \dfrac{ker(\overline{\partial}_q)}{\overline{\partial}_{q-1}(\mathcal{E}^{p,q-1})}$.

Needless to say by using ∂ instead of $\overline{\partial}$ we get a similar cochain with the first index rising instead, but this one is somewhat less interesting; the reason is the fact that the kernel of $\overline{\partial} : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1}$ will turn out to be the sheaf Ω^p of *holomorphic p-forms* that is the sheaf of forms of degree p of type $\sum_I f_I dz_I$ with f_I holomorphic. This will lead us to consider the above sequence as a natural resolution of Ω^p; we will see this in detail in the next chapter.

1.3 Locally Convex Spaces

We will go through some of the theory of locally convex (complex) topological vector spaces that is involved in the proof of Serre Duality. A *topological vector space* is a vector space endowed with a topology that makes the algebrical operations continuous. A subset $Y \subset X$ is called

- *convex* if for all $x, y \in Y$ and $0 \leq \lambda \leq 1$ the line $\lambda x + (1-\lambda)y$ is in Y;
- *balanced* if whenever $\lambda \in \mathbb{C}$ and $|\lambda| \leq 1$ then $\lambda Y \subseteq Y$;
- *absorbing* if $\forall x \in X$ exists $\lambda > 0$ such that $x \in \lambda Y$.

Note that both absorbing and balanced sets contain 0. A topological vector space over $\mathbb{C}$ is said to be *locally convex* if there exist a open neighbourhood system of 0 whose elements are all convex sets; a locally convex, metrizable, complete space is called a *Fréchet* space.

Locally convex topological structures can be constructed via a certain class of functionals, the *seminorms.* A seminorm over a (complex) topological vector space X is a continuous function $\rho : X \to \mathbb{R}$ such that

i) $\rho(x+y) \leq \rho(x) + \rho(y),\ \forall x, y \in X$;

ii) $\rho(\lambda x) = |\lambda|\rho(x),\ \forall \lambda \in \mathbb{C}, x \in X$.

An example of seminorm is the *Minkowski functional* of a convex absorbing and balanced subset $Y \subset X$ defined as

$$\mu_Y(x) = \{\inf \lambda > 0 |\ x\lambda^{-1} \in Y\};$$

it is important to remark that elements of Y are characterized in terms of the Minkowski functional of Y as those $x \in X$ such that $\mu_Y(x) < 1$. By using seminorms we define neighbourhoods systems of 0 which by translating form neighbourhood systems of any point on X so that a neighbourhood system of 0 completely defines a topology on X.

Proposition 1.3.1. *If ρ is a seminorm on a vector space X then the ρ-ball*

$$B_{\rho,r}(0) = \{x \in X |\ \rho(x) < r\}$$

centered at 0 of radius r is convex, balanced and absorbing.

Proof. If $0 \leq \lambda \leq 1$ and $x, y \in X$ then $\rho(\lambda x + (1-\lambda)y) \leq |\lambda|\rho(x) + |1-\lambda|\rho(y) \leq r$; if $|\lambda| \leq 1$ it is $\rho(\lambda x) = |\lambda|\rho(x) \leq r$ and finally for $x \in X$ take $\lambda > \rho(x)$ so that $(r/\lambda)\rho(x) < r$ which means $x \in (\lambda/r)B_{\rho,r}(0)$ □

Using the proposition 1.3.1 we can generate a topology on X by means of a collection of seminorms as illustrated by the following

Theorem 1.3.2. *Let X be a vector space and $\mathcal{P} = \{\rho_i\}_{i \in I}$ a collection of seminorms on X. Then X has a locally convex topological vector space structure in which each ρ_i is continuous.*

Proof. Our local basis $\mathcal{B}$ at 0 is made by the collection of all *finite* intersections $\bigcap_n B_{i,r}(0)$ of open balls $B_{i,r}(0) = \{x \in X | \rho_i(x) < r\}$ which by the previous proposition are balanced, convex absorbing neighbourhoods of 0. Moreover all sets in $\mathcal{B}$ are convex, balanced and absorbing being immediate that all these properties are mantained under intersection and homotethy.

Declare a neighbourhood U of any $x \in X$ to be a subset of X with the property $U \supset x + B$ for a certain $B \in \mathcal{B}$; this produces a topology for which by construction $\mathcal{B}$ is a local basis at 0 of convex sets. In this topology every ρ_i is continuous because by definition inverse images of open sets in $\mathbb{R}$ are open neighbourhoods of 0. We just have to prove continuity of the algebraic operations. In topological vector spaces this is equivalent to the following: the sum is continuous if for any neighbourhood U of $x + y$ in x there exist neighbourhoods V of x and W of y such that $V + W \subset U$; the scalar multiplication is continuous if for all $x \in X$ and $\lambda \in \mathbb{C}$ if $\lambda x \in U$ then there are $\epsilon > 0$ and a neighbourhood $V \ni x$, such that whenever $|\mu - \lambda| < \epsilon$ then $\mu V \subset U$.

For $x, y \in X$ and for any $B \in \mathcal{B}$, $x + \frac{1}{2}B$, $y + \frac{1}{2}B$ are in $\mathcal{B}$ and $x + \frac{1}{2}B + y + \frac{1}{2}B = (x+y) + B$ so addition is continuous. Fix $B \in \mathcal{B}$ and $\lambda > 0$; let, by 'absorbing', ϵ such that $\epsilon x \in \frac{1}{2}B$, and $y \in x + \frac{1}{2(|\lambda|+\epsilon)}B$. We just have to show that $\mu y - \lambda x \in B$ whenever $|\mu - \lambda| < \epsilon$. But

$$\mu y - \lambda x = \mu(y - x) + (\mu - \lambda)x;$$

the second term is in $\frac{1}{2}B$ by the choice of ϵ and the first is in $\frac{\mu}{2(|\lambda|+\epsilon)}B$; but $\mu/(|\lambda| + \epsilon) < 1$ and $\frac{1}{2}B$ is balanced so that $\mu(y - x) \in \frac{1}{2}B$. □

A familiy $\{\rho_i\}$ of seminorms on X is said to be *separating* if whenever $x \neq 0$ there exists i such that $\rho_i(x) \neq 0$. A separating family of seminorms generates a Hausdorff topology since if $x \neq y$ then $\exists \rho_i$ such that $\rho_i(x - y) = r > 0$ thus $x + B_{i,r/2}$ and $y + B_{i,r/2}$ separate x and y.

Conversely suppose X Hausdorff and locally convex and take a local basis $\mathcal{B}$ at 0. By substituting each element $B \in \mathcal{B}$ with the union of all the balanced subsets contained in B (which is balanced) we may choose $\mathcal{B}$ consisting of balanced sets. Since every set in a neighbourhood basis at 0 is absorbing we see that the Minkowski functionals μ_B for $B \in \mathcal{B}$ are seminorms, which define $\mathcal{B}$ by the characterization of the elements of $\mathcal{B}$ via μ_B. Being X Hausdorff if $x \neq 0$ there exists B not containing x, therefore $\mu_B(x) \geq 1$ so that $\{\mu_B, B \in \mathcal{B}\}$ is a separating family of seminorms generating $\mathcal{B}$. We add up everything in the following theorem

Theorem 1.3.3. *A locally convex space X is metrizable if and only if it is Hausdorff and has a countable local basis.*

Proof. If X is metrizable then of course it is Hausdorff and $D_n = \{x \in X| \ d(0, x) < \frac{1}{n}\}$ is a countable local basis of neighbourhoods of 0. On the other hand let $\mathcal{B} = \{B_n\}$ is a countable local basis for a Hausdorff space X; by the above its topology is generated by the Minkowski functionals $\mu_n = \mu_{B_n}$. Set

$$d(x,y) = \sum_n \frac{\mu_n(x-y)}{2^n(1+\mu_n(x-y))}$$

which is well defined because the sum converges. d is subadditive; indeed for all n

$$\begin{aligned}\frac{\mu_n(x-y)}{1+\mu_n(x-y)} &= \frac{\mu_n(x-z+z-y)}{1+\mu_n(x-z+z-y)}\\ &\leq \frac{\mu_n(x-z)+\mu_n(z-y)}{1+\mu_n(x-z)+\mu_n(z-y)}\\ &\leq \frac{\mu_n(x-z)}{1+\mu_n(x-z)} + \frac{\mu_n(z-y)}{1+\mu_n(z-y)}\end{aligned}$$

Furthermore d is symmetric since $\rho_n(x-y) = \rho_n(-(y-x)) = \rho_n(y-x)$ and $x = y$ implies $\mu_n(x-y) = 0$ for all n so $x = y$ since $\{\mu_n\}$ is separating. So d is a metric. We must check that induces the same topology of that generated by the μ_ns; for this is sufficient to exhibit a new local basis $\mathcal{U}$ of neighbourhoods of 0 made by metric open sets.

Consider $U_n = \{x \in X |\, d(0,x) < 2^{-n}\}$. These are open sets, in fact d is the limit of a uniformly convergent series of continuous functions, thus itself continuous; the inverse image U_n of $(-\infty, 2^{-n})$ under d must then be open. Finally $U_{n+1} \subset B_n$ because if $x \notin B_n$ then $\mu_n(x) \geq 1$ whence

$$d(0,x) = \frac{\mu_n(x)}{2^n(1+\mu_n x)} \geq \frac{1}{2^{n+1}}$$

so that $x \notin U_{n+1}$. □

To conclude this section we prove that Fréchet spaces are well-behaved with respect to the quotient by a closed subspace

Proposition 1.3.4. *Let X be a Fréchet space, $C \subset X$ a closed subspace. Then X/C is also a Fréchet space.*

Proof. Let $\{B_n\}$ be a countable local basis of 0; $\{\pi(B_n)\}$ contains a countable local basis of X/C and if each B_n is convex so is its image under the projection. Thus by theorem 1.3.3 to prove that X/C is metrizable all we have to do is check that X/C is Hausdorff: this is the case being C closed. Consider the continuous map

$$\begin{aligned}\lambda : X \times X &\longrightarrow X\\ (x,y) &\mapsto x-y\end{aligned}$$

the inverse image of C under λ is the graph $\mathcal{G} = \{(x,y) |\, x-y \in C\}$ of the equivalence relation associated with C, which is therefore closed. Let π^2 be the quotient map: $\pi^2 : X \times X \to \frac{X \times X}{C \times C}$; it is itself a topological identification with respect to which $\mathcal{G}$ is saturated. This means that $\pi^2(\mathcal{G}) = \{[x,x], x \in X\}$ is closed. Now suppose that the canonical projection π is open; then the bijective homomorphism $\phi : \frac{X\times X}{C\times C} \to \frac{X}{C} \times \frac{X}{C}$ has a continuous inverse and hence is a homeomorphism. π is indeed open because for every open subset $U \subset X$ the canonical image of its saturation $\pi^{-1}(\pi(U))$ is by definition open and

$$\pi^{-1}(\pi(U)) = \pi^{-1}(\{[x]_C, x \in U\}) = \bigcup_{y\in C} y + U$$

is open because the translation is open. Hence we conclude that $\phi(\pi^2(\mathcal{G}))$ is homeomorphic to its image $\Delta_\pi = \{([x],[x]), x \in X\}$, the diagonal of $\frac{X}{C} \times \frac{X}{C}$ which is then closed; then if $[x] \neq [y]$ there exists an open neighbourhood $V \times W$, $[x] \in V$, $[y] \in W$, V and W open in X/C such that $(V \times W) \cap \Delta_\pi = \emptyset$ so that $V \cap W = \emptyset$, and we have proved that X/C is Hausdorff.

To show the completeness choose a sequence $\{U_n\}$ of saturated neighbourhoods of 0 in X in such a way that $2U_n \subset U_{n-1}$ and suppose that $[u_n]$ is a Cauchy sequence of cosets in the quotient. If $[u_m]-[u_n] \subset U_{n_0}$ for $m > n > n_0$ and a certain saturated open set U_{n_0}, all $y \in [u_m]$, $x \in [u_n]$ are such that $y - x \in U_{n_0} + C$. But $y \in x + U_{n_0} + C$ implies $[u_m] \cap x + U_{n_0} \neq \emptyset$ for all $x \in [u_n]$. This means that we can select a sequence $\{x_n\}$ in X such that $x_n \in [u_n]$ for every n in such a way that $x_{n+1} \in x_n + U_n$ whence, by induction, $\forall n > n_0$, $p > 0$

$$x_{n+p} \in x_n + (U_{n+p-1} + \ldots + U_n) \subset x_n + \sum_{j=0}^{p} U_n \subset x_n + 2U_n \subset x_n + U_{n-1},$$

whence $x_{n+p} - x_n \in U_{n_0}$. We have thus obtained a Cauchy sequence $\{x_n\}$ in X which by completeness converges to x; but then $x_n + c \to x + c$ for all $c \in C$ implies $[u_n] \to [x]$ in X/C. $\square$

1.4 Currents

Currents are linear continuous functionals from the complex topological vector space $\mathcal{E}^{p,q}$ with respect to a certain topology that makes it into a complete space. This topology has been introduced by Schwartz in [21], the work in which he founded distribution theory; the currents are just an extension of the idea of distribution to the space of differential forms.

M is always a complex analytic manifold of dimension n. We begin from the notion of support for forms, entirely analogue to that of a function; the *support* of a k-differential form ω on M is the closure of the set where $\omega \neq 0$. For each U the set $\mathcal{E}^k_*(U)$ of k-differential forms having compact support in U are vector subspaces of $\mathcal{E}^k(U)$; with U varying in M the correspondence $U \mapsto \mathcal{E}^k_*(U)$ is a sheaf $\mathcal{E}^k_*$, subsheaf of $\mathcal{E}^k$.

Define, for a multi-index $J = (j_1, \ldots, j_m)$ of length m arbitrary (possibly with repeated indices), ∂^J to be the partial derivative with respect to the coordinates $z_{j_1}, \ldots, z_{j_m}$. Fix a compact set $K \subset U \subset M$ and let $\mathcal{E}^k_K(U)$ be the space of k-differential forms with support in

K. We denote by τ_k the topology generated by the family of seminorms $\rho_{K,J}$, depending on J, given by

$$\rho_{K,J}(\omega) = \sup_I \sup_{z \in K} |\partial^J f_I(z)|$$

where f_I indicates a local expression on a coordinate covering of K of the I-th coefficient of ω. To be more precise the sup should also keep track of the fact that may be different local expressions of f_I in z. But since K is compact we can choose a finite number of local charts covering K, therefore at most a finite number of local expressions for f_I are given on K, and the supremum is well defined.

This family of seminorms is indeed countable and separating so by theorem 1.3.3 the induced topology is metrizable and yields a locally convex space. It is a simple consequence of theorems on uniform convergence the fact that this topology is complete (we will discuss this further in section 2.2). This means that $\mathcal{E}^k_K(U)$ is a Fréchet space: its topology is indeed that of *uniform local convergence* of every derivative.

If we now let K vary on every possible compact subset of M, the family $\rho_{K,J}$ becomes defined on the whole $\mathcal{E}^k_*(U)$; so it may seem reasonable to consider on $\mathcal{E}^k_*(U)$ the topology induced by the family of seminorms above, depending on K. Unfortunately this is not a good choice the reason being that the resulting topology will be still metrizable but not complete (nevertheless this topology will be used later in section 2.2; in that situation there will be no requirements on supports, therefore no issues of convergence). For instance let $U = \mathbb{C}$, $k = 0$ and f be a smooth function with support in $D_1(0)$; then

$$\psi_n(x) = \sum_{m=1}^{n} \frac{1}{m} f(z - m) \tag{1.5}$$

is a Cauchy sequence. To see this let $\{K_n\}$, be a sequence of compacts in M with $K_n \subset K_{n+1}$ (which exists being M metrizable); a local basis can be choosen by $V_J = \{\omega \in \mathcal{E}^0_K(\mathbb{C}) | \rho_{k,J}(\omega) < 1/J\}$. Then for $J > 0$ there is a certain $n(J)$ such that for $n_1, n_2 > n(J)$

$$\left| \sum_{m=n_1}^{n_2} \frac{1}{m} \partial^J f(z - m) \right| < \frac{1}{J}$$

and $\psi_{n_1} - \psi_{n_2}$ is in V_J; but $\lim_{n\to\infty} \psi_n(z)$ does not have compact support, so it does not converge in $\mathcal{E}^0_*(\mathbb{C})$.

The right choice to make $\mathcal{E}^k_*(U)$ into a complete space is taking the *finest locally convex topology* τ that makes every inclusion map $i_K : \mathcal{E}^k_K(U) \hookrightarrow \mathcal{E}^k_*(U)$ continuous. Equivalently τ is described as follows: declare a neighbourhood of 0 to be a convex balanced set V such that for each compact K the intersection $V \cap \mathcal{E}^k_K$ is a neighbourhood of 0 in τ_k.

The class of all such sets is a local basis for 0 and generates τ by means of translation and arbitrary union. We remark that the subspace topology of $\mathcal{E}^k_K(U)$ inherited from τ is precisely

τ_k: let $V \in \tau_k$ and for E an open set in τ_K; for $\omega \in E$ let δ, J be such that

$$B_{\delta,J}(\omega) = \{\eta \in \mathcal{E}^k_K(U) | \rho_{J,K}(\omega - \eta) < \delta\} \subset E.$$

Consider in τ the open neighbourhood of 0

$$W_{\delta,K,J} = \{\eta \in \mathcal{E}^k_*(U) | \rho_{J,K}(\eta) < \delta\} :$$

it is

$$\mathcal{E}^k_K(U) \cap (\omega + W_{\delta,K,J}) = \omega + (\mathcal{E}^k_K(U) \cap W_{\delta,K,J}) = B_{\delta,J}(\omega) \subset E$$

so the open set $V = \bigcup_{\omega,J,K,\delta} \omega + W_{\delta,K,J}$ is such that $V \cap \mathcal{E}^k_K(U) = \bigcup_{\delta,\omega,J} B_{\delta,J}(\omega) = E$. Convergence in this topology is characterized as follows

Proposition 1.4.1. *A sequence $\omega^n \in \mathcal{E}^k_*(U)$ converges to 0 in τ if and only if*

1. *There exists $K \subset U$ compact such that Supp$\omega^n \subset K$ for all n;*
2. *$\partial^J f_I \to 0$ uniformly on K for all I, J.*

Proof. If 1 and 2 hold we are just saying that $\omega^n \to 0$ in $\mathcal{E}^k_K(U)$ with τ_k; but inclusion is continuous so $\omega^n \to 0$ in τ. Conversly assume that ω^n converges in τ. To prove 1 suppose there is no such a K. Then if f^n_I are local coefficients of ω^n in U we can find a sequence $x_n \in$ Suppω^n without any accumulation point in U such that $f^n_I(x_n) \neq 0$. Take the set

$$V = \{\omega \in \mathcal{E}^k_*(U), |f_I(x)| < |f^n_I(x_n)|, \forall n \in \mathbb{N}, x \in U\};$$

where f_I are the coefficients of ω: V it is not reduced to 0 since x_n has no accumulation point. Also, it is such that $V \cap \mathcal{E}^k_K(U)$ is a neighbourhood of 0 for all compact set K, the reason being that since x_n does not accumulate, every K contains at most a finite number of x_n, so the coefficients of ω are uniformly bounded. Therefore V is a neighbourhood of 0 in τ but we then see that $\omega^n \notin V$ for every n, which says that ω^n does not converges to 0.

Now if ω^n is commonly supported in a compact K and converges to 0 in τ, it also converges to 0 in τ_k because τ induces τ_k in $\mathcal{E}^k_K(U)$; this means that every for every ϵ_J there exist $n_{J,\epsilon}$ such that for $n > n_{J,\epsilon}$ we have $\sup_{z \in K} |\partial^J f^n_I(z)| < \epsilon_J$ for all I, which amounts to say that the J-th derivative of the coefficients converges to 0 uniformly on K. □

Remark. We remark that τ is not metrizable. To see this note that every $\mathcal{E}^k_K(U)$ is complete in τ_k, therefore complete in the induced topology by τ, thus closed in $\mathcal{E}^k_*(U)$. Clearly it has also empty interior; if that were not the case then if $V \in \tau$ were a neighbourhood of 0 such that $V \subset \mathcal{E}^k_{K_0}$ for a certain compact K_0, this would imply $\mathcal{E}^k_{K_0} \cap \mathcal{E}^k_K \neq 0$ for every compact K, which is impossible. By choosing a sequence K_n of compact sets such that $U \subseteq \bigcup_n K_n$ we see that $\mathcal{E}_*^k(U) = \bigcup_n \mathcal{E}_{K_n}^k(U)$; each $\mathcal{E}_{K_n}^k$ is complete, therefore by Baire Lemma the only possibility left is that $(\mathcal{E}_*^k(U), \tau)$ is not metrizable.

By theorem 1.3.3 this implies that $\mathcal{E}_K^k$ with τ is not even first-countable, not having a countable local basis. Therefore it is a remarkable result that a sequencially continuous function $f : \mathcal{E}_K^k(U) \to Y$ to a locally convex topological vector space Y(i.e. a function bringing converging sequences in $\mathcal{E}_K^k$ to converging sequences in Y) is also continuous (see [20] chap. 6 theorem 6.6).

Definition 1.4.2. A *current* of degree $k \leq n$ on an open set U of a manifold M of dimension n is a continuous linear mapping $T : \mathcal{E}_*^{n-k}(U) \to \mathbb{C}$ with respect to τ. A current of degree n is called a *distribution* on U and is just a continuous linear functional from $C_*^\infty(U)$. The set of all such functionals is denoted $\mathcal{D}^k(U)$

In other words a current T of degree k is just an element of the topological dual $\mathcal{E}_*^{n-k}(U)^*$ of $(\mathcal{E}_*^{n-k}(U), \tau)$ and has therefore the natural topological structure of the weak-* topology. We recall that this is the coarsest topology on $\mathcal{E}_*^{n-k}(U)^*$ that makes continuous every element of the algebraic bidual space $\mathcal{E}_*^{n-k}(U)^{**}$ in the form $\omega(T) = T(\omega)$, for every k-currents T and for a certain $n-k$ form ω with compact support in U. Converging sequences in the weak-* topology are characterized by

$$T_h \to 0 \Leftrightarrow T_h(\omega) \to 0 \ \ \forall \omega \in \mathcal{E}_*^{n-k}(U).$$

There are some very well known examples of currents and distributions. For instance on $U \subset M$ fix a form support ω of total degree k. For every compactly supported $n-k$-form η on U the mapping $\eta \mapsto \int_U \omega \wedge \eta$ is well defined being M orientable and having η compact support; continuity can be seen by passing the limit under the integral sign. Thus we can associate to each $\omega \in \mathcal{E}_*^k(U)$ a current T_ω of degree k and consequently we can view $\mathcal{E}^k(U)$ as a subset of $\mathcal{D}^k(U)$. A remarkable example of current that is not an integral is the so called *Dirac distribution* $\delta_p : C^\infty(U) \to \mathbb{C}$ which sends a smooth function f to its value in p.

Currents of the same degree form a vector space on $C^\infty(U)$; for $\omega \in \mathcal{E}_*^{n-k}(U)$ we set, for currents T and S, the current $(T+S)(\omega) = T(\omega) + S(\omega)$ and $gT(\omega) = T(g\omega)$. We denote this space by $\mathcal{D}^k(U)$.

Also a wedge product between a current and a form is defined; if T is a current of degree k and ω a form of degree p, $T \wedge \omega$ is a $k+p$ current defined by

$$(T \wedge \omega)(\eta) = T(\omega \wedge \eta)$$

which reduces to the scalar product in the case ω has degree 0 and to the ordinary wedge product between forms if $T = T_\omega$. As for forms this product is skew-symmetric with $T \wedge \omega = (-1)^{pk} \omega \wedge T$.

A notion of *support* is also defined for currents; we say that $T \equiv 0$ in an open set U if $T(\omega) = 0$ for all ω supported in U. We define the support of T to be the closure of the complementary of the maximal open set U where $T \equiv 0$. A current is *compactly supported* if its support is compact. The set of currents of degree k with compact support contained in U is denoted $\mathcal{D}^k_*(U)$; the sheaf $\mathcal{D}^k_*$ is a subsheaf of $\mathcal{D}^k$.

If T is a current of degree k with compact support we can extend it to a continuous linear functional on the whole $\mathcal{E}^k$ as follows. Take a C^∞ partition of unity $\{\rho_\alpha\}$ subordinated to a locally finite covering $\{U_\alpha\}$ of M. If $\omega \in \mathcal{E}^k$ is such that the series $\sum_\alpha T(\rho_\alpha \omega)$ converges for every ρ_α then the sum equals $T(\omega)$ by the properties of the partition of unity. If $T \in \mathcal{D}^k$ has compact support then only a finite number of summands are nonzero and the sum always converges; therefore we have extended T on the whole $\mathcal{E}^k$. When we deal with $\mathcal{D}^k_*$ we implicitly mean compactly supported currents as functionals extended on the whole of $\mathcal{E}^k$.

For compactly supported distributions integration is defined by declaring $\int_M T = T(1)$: this coincides with the usual definition of integration of n-forms if $T = T_\omega$ for some n-form ω. Note that if the distribution is of the form $\omega \wedge T$ for a $n-k$ form ω and a k-current T then $\int_M \omega \wedge T = T(\omega)$.

Another useful fact is that locally currents have a simple description as collections of distributions. Take a current of degree k and let $\omega = \sum_I f_I dx_I$ be a local expression of a $n-k$ form with compact support. For any multi-index I of length $n-k$ we set the 'coordinates' T^I of T to be the distributions $T^I = T \wedge dx_I$ sending a smooth function f to $T(f dx_I)$.

By linearity we have

$$T(\omega) = \sum_I T^I e_I(\omega)$$

where e_I is the operator sending ω to the coefficient f_I of its I-th summand $f_I dx_I$. This yields a decomposition

$$T = \sum_I T^I e_I.$$

These local expressions are compatible on intersections of local charts thanks to the coordinate changes on M; this shows that $\mathcal{D}^k(U)$ is locally a direct sum of $\binom{n}{n-k}$ copies of $\mathcal{D}^n(U)$.

A differential operator for currents, also called d must be now introduced. If $T \in \mathcal{D}^k(U)$ we let $dT \in \mathcal{D}^{k+1}(U)$ be

$$dT(\omega) = T(d\omega) \quad \forall \omega \in \mathcal{E}^{n-k-1}$$

As for the differential on forms it is

$$d(T \wedge \omega) = dT \wedge \omega + (-1)^k T \wedge d\omega$$

and also it is easy to see that $dT_\omega = (-1)^{k+1} T_{d\omega}$.

Proposition 1.4.3. *Let M be a complex analytic manifold. For $U \subset M$ the correspondence $U \to \mathcal{D}^k(U)$ is a sheaf of C^∞-modules over M.*

Proof. $\mathcal{D}^k$ is made into a presheaf in the following way: for $T \in \mathcal{D}^k(U)$ define its restriction $T|_V$ to $V \subset U$ by $T|_V(\omega) = T(\omega)$ for all $\omega \in \mathcal{E}^k_*(U)$ such that $\mathrm{Supp}\omega \subset V$. The module structure has been previously outlined. Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite covering of M and S, T currents on $U \subset M$ agreeing on $U_\beta \cap U_\alpha$.

Choose a C^∞ partition of unity $\{\rho_\alpha\}$ subordinated to $\mathcal{U}$. Then for $\omega \in D^k(U)$ we have $\omega = \sum_\alpha \rho_\alpha \omega$ and there are no issues of convergence being $\mathcal{U}$ locally finite; thus

$$T(\omega) = \sum_\alpha T(\rho_\alpha \omega) = \sum_\alpha S(\rho_\alpha \omega) = S\left(\sum_\alpha \rho_\alpha \omega\right) = S(\omega)$$

so that the first axiom is checked. Now select a collection of currents $\{T_\alpha\}$ in $D^k(U_\alpha)$ for each α such that $\forall \alpha, \beta$, it is $T_\alpha|_{U_\beta} = T_\beta|_{U_\alpha}$. If ω is a differential form whose support is in U_β we have that $\mathrm{Supp}\{\rho_\alpha \omega\}$ is in $U_\alpha \cap U_\beta$ for all α. We set $T(\omega) = \sum_\alpha T_\alpha(\rho_\alpha \omega)$ so that $T_\alpha(\rho_\alpha \omega) = T_\alpha|_{U_\beta}(\rho_\alpha \omega)$; using this fact and the hypothesis

$$T(\omega) = \sum_\alpha T_\alpha|_{U_\beta}(\rho_\alpha \omega) = \sum_\alpha T_\beta|_{U_\alpha}(\rho_\alpha \omega) = \sum_\alpha T_\beta(\rho_\alpha \omega) = T_\beta(\omega).$$

Finally T is continuous because if $\omega^i \to 0$ then $T_\alpha(\omega^i) \to 0$ for all α so that $T(\omega^i) \to 0$; therefore T is actually a current. □

Naturally arises a cochain complex of sheaves

$$0 \to \mathcal{D}^0 \xrightarrow{d} \mathcal{D}^1 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{D}^n \to 0$$

where $d : \mathcal{D}^k \to \mathcal{D}^{k+1}$ is given on open sets by the differential operator on currents; its square is 0 and it commutes with restrictions because the differential on forms does.

Let us look again at the decomposition

$$\mathcal{E}^k_M = \bigoplus_{p+q=k} \mathcal{E}^{p,q}.$$

If we take sheaves of compactly supported forms instead of forms with arbitrary support this decomposition holds as well. Now we call *current of type* (p, q) a continuous linear mapping from the topological vector space $\mathcal{E}^{n-p,n-q}_*(U)$ to $\mathbb{C}$ in the topology τ. The set of these functionals forms a (topological) subspace $\mathcal{D}^{p,q}(U)$ of $\mathcal{D}^k(U)$. Each functional on $\mathcal{E}^{n-k}_*(U)$ can be written uniquely as the sum of its restrictions to $\mathcal{E}^{n-p,n-q}_*(U)$; hence we have an induced decomposition of sheaves

$$\mathcal{D}^k = \bigoplus_{p+q=k} \mathcal{D}^{p,q}$$

Again by setting $\overline{\partial} = \pi_{p,q+1} \circ d$, $\partial = \pi_{p,q+1} \circ d$ (whose square is 0 just as for ∂ and $\overline{\partial}$ in the Dolbeault complex) a corresponding cochain complex of sheaves

$$0 \to \mathcal{D}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{D}^{p,1} \dots \xrightarrow{\overline{\partial}} \mathcal{D}^{p,n} \to 0. \tag{1.6}$$

is established, along with his analogous for ∂.

1.5 Regularization and Laplacian on Currents

In the first part of this section we will briefly summarize the formal properties of the regularization (or smoothing) of currents on the n-complex space via compactly supported smooth functions; next we introduce the Laplacian operator on currents on arbitrary manifolds, and outline its relations with the $\overline{\partial}$ operator. These ideas will be used in the next chapter to prove regularity results concerning the Laplace equation on distributions.

With reference to the notation of section 1.4 a distribution T on $\mathbb{C}^n$ is an element of $\mathcal{D}^{n,n}$, that is, a linear continuous functional on C^∞; if $dz = dz_1 \wedge \dots \wedge dz_n \wedge d\overline{z}_1 \wedge \dots \wedge d\overline{z}_n$ then $T = T_\omega$ for a (n, n) form $\omega = f dz$ with compact support, if

$$T(g) = \int_{\mathbb{C}^n} g(z) f(z) dz.$$

Distributions of this kind are called *regular*. Since every (n, n) form on $\mathbb{C}^n$ can be identified with its coefficient there is a perfect correspondence between the spaces $\mathcal{D}^{n,n}(\mathbb{C}^n)$ and $\mathcal{D}^{0,0}(\mathbb{C}^n)$. For simplicity, in everything that follows we will write a regular distribution T_f for a smooth function f, instead of T_ω for an (n, n) form ω. In view of the previous claim this is consistent and should be not misleading. From the stand-alone point of view of this chapter it is therefore possible to consider a distribution T on $\mathbb{C}^n$ indifferently as an element of $\mathcal{D}^n$ or $\mathcal{D}^0$; however when cohomology comes in it is important to understand distributions as the set $\mathcal{D}^0$ because we are interested in the kernel of the first morphism of 1.6. This complication arises because we need to give a *cochain* complex structure to the space of currents; if we instead defined a (somewhat more natural) *chain* complex structure this would not have occurred.

We show now that every distribution can be 'smoothed', which means that can be considered as a limit of smooth function.

Let $\chi(z) \in C^\infty(\mathbb{C}^n)$ be a compactly supported function whose integral over $\mathbb{C}^n$ is 1 and such that $\chi(z) = \chi(|z|)$, i.e. it is radially symmetric. For each $\epsilon > 0$ set

$$\chi_\epsilon(z) = \frac{1}{\epsilon^n} \chi\left(\frac{z}{\epsilon}\right).$$

Of course $\int_{\mathbb{C}^n} \chi_\epsilon(z) dz$ is still 1. For a distribution T the regularization of T by the kernel χ_ϵ is the C^∞ function

$$T_\epsilon(z) = T(\chi_\epsilon(z - w))$$

where obviously T acts on the variable w. As we will see shortly this is nothing but a generalization of the convolution product between functions (one with compact support), defined by $(f * g)(z) = \int_{\mathbb{C}^n} f(w)g(w-z)dw$. With an abuse of notation we still use T_ϵ for the distribution

$$T_\epsilon(g) = \int_{\mathbb{C}^n} g(z)T_\epsilon(z)dz.$$

Similarly for a function f and χ_ϵ as before we define $f_\epsilon := f * \chi_\epsilon$. We denote by $C^\infty_*(\mathbb{C}^n)$ the space of differentiable functions on $\mathbb{C}^n$ with compact support. The regularization product on distributions obeys to the following rules

1. $(T_f)_\epsilon = T_{f_\epsilon}$ for every $f \in C^\infty(\mathbb{C}^n)$;
2. $T_\epsilon(g) = T(g_\epsilon)$ for every $g \in C^\infty_*(\mathbb{C}^n)$;
3. $(\partial^J T)_\epsilon = \partial^J(T_\epsilon)$ for any multi-index J.

that are straightforwardly checked. We remark that for a function with compact support we have, by usual properties of convolution product that $\partial^J(f_\epsilon) = (\partial^J f)_\epsilon \to \partial^J f$, as ϵ approaches 0, uniformly for each derivative ∂^J, and by property 2 this means that $T_\epsilon \to T$ in the weak-* topology. In fact by definition of τ

$$\begin{aligned} &f_\epsilon \to f \Rightarrow \partial^J f_\epsilon \to \partial^J f \text{ uniformly for all } J \\ &\Rightarrow T_\epsilon(\partial^J f) = T((\partial^J f)_\epsilon) \to T(\partial^J f) \end{aligned}$$

and thus $T_\epsilon \to T$.

We can transfer all of the above to (p, q) currents as follows. We write, accordingly with section 1.4, T as a formal sum of distributions relative to the decomposition of $\mathcal{D}^k = \bigoplus_{p,q} \mathcal{D}^{p,q}$. It is $T = \sum_{I,J} T^{IJ} e_{I,J}$, where $e_{I,J}$ is the functional sending the summand of ω relative to the differential of I holomorphic coordinates and J complex conjugate coordinates in its coefficient. We then set $T_\epsilon = \sum_{I,J} T^{IJ}_\epsilon e_{I,J}$. Again we have $T_\epsilon \to T$ in the topology of $\mathcal{D}^{p,q}$ and property 3 by using linearity translates into $\overline{\partial}(T_\epsilon) = (\overline{\partial} T)_\epsilon$.

Now we explain some ideas from harmonic theory on compact manifolds. These will come into use in the next chapter when proving regularity results of harmonic distributions.

Let M be an analytic compact connected complex manifold; the generalization of the Laplacian operator is related to a hermitian structure h given on M. In this circumstance the Laplacian is in fact defined by means of the adjoint operator $\overline{\partial}^*$ of $\overline{\partial}$ with respect to a certain Hilbert space structure of $\mathcal{E}^{p,q}_*(M)$ given by means of h. The following brief discussion summarizes how this structure comes into play.

A *hermitian metric* h is a smooth assignment to every $z \in M$ of a hermitian product $<,>_z$ on $T_zM \times \overline{T_zM}$. Here T_zM is the holomorphic tangent space at z and $\overline{T_zM}$ its conjugated: this means, recalling the isomorphism between the space of bilinear forms from $T_zM^\times\overline{T_zM}$ to $\mathbb{C}$ and

$T_z^*M \otimes \overline{T_z^*M}$, that h is a section of the bundle $T^*M \otimes \overline{T^*M}$. We can therefore give an expression of h in a local frame $dz_1, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n$ as a tensor $\sum_{i,j=1}^n h_{i,j} dz_i \otimes d\overline{z}_j$, for certain smooth functions $h_{i,j}$ that are the entries of the hermitian matrix defining $<,>_z$ at each point. It turns out that there exists a global (n,n) volume form vol_M canonically associated with h; canonically in the sense that vol_M is obtained by the $(1,1)$ form $\omega = \frac{i}{2} \sum_{i,j=1}^n h_{i,j} dz_i \wedge \overline{d} z_j$, the *fundamental form of* h, that completely defines h.

Imagine now that h is locally given by an orthonormal frame at each z, that is, in the basis $dz_i, \ldots, dz_n, d\overline{z}_1, \ldots, d\overline{z}_n$ the matrix $h_{i,j}$ is the identity: this can always be done by means of the Graham-Schmidt process. Then $<,>_z$ carries over to a bilinear form on $\Lambda^{p,q}(T^*M)$ simply by setting the basis vectors $\{dz_I(z) \wedge d\overline{z}_J(z)\}_{|I|=p,|J|=q}$ to be orthonormal and of squared length 2^{p+q}. We still denote this product $<,>_z$.

In conclusion we then have a global inner product on M given by integrating the form $<,>_z$ with respect to vol_M(for details refer to [10] chap. 0, sec.6, p. 80, and [27] chap. 3, p.65):

$$(\omega, \eta) := \int_M < \omega(z), \eta(z) >_z vol_M(z) \quad , \quad \forall \omega, \eta \in \mathcal{E}^{p,q}(M)$$

This product makes $\mathcal{E}^{p,q}(M)$ into a pre-Hilbert Space which is actually complete with respect to $(,)$, so it is actually a Hilbert space. Moreover $\overline{\partial} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q+1}(M)$ turns out to be a bounded linear operator according to the norm induced by $(,)$ thus it has an adjoint $\overline{\partial}^* : \mathcal{E}^{p,q+1}(M) \to \mathcal{E}^{p,q}(M)$. We define the *Laplacian* Δ of a (p,q) form to be the operator $\Delta := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{p,q}(M)$.

Now we would like to give an explicit description of $\overline{\partial}^*$. For that purpose we need to use the so called *Hodge duality operator* $* : \mathcal{E}^{p,q}(M) \to \mathcal{E}^{n-p,n-q}(M)$ that sends ω to the only $*\omega$ such that for every $z \in M$, $\eta \in \mathcal{E}^{p,q}(M)$

$$(\omega(z), \eta(z)) vol_M(z) = \eta(z) \wedge *\omega(z).$$

In local orthonormal coordinates if $\omega = \sum_{I,J} f_{I,J} dz_I \wedge d\overline{z}_J$ then $*\omega$ is explicitely given by

$$*\omega = 2^{p+q-n} \sum_{I,J} sgn(\sigma_{I,J}) \overline{f}_{I,J} \phi_{I^c} \wedge \phi_{J^c}$$

where complementaries are in the set $\{1, \ldots, n\}$, $\sigma_{I,J}$ is the permutation of $2n$ elements $(1, \ldots, n, 1, \ldots, n) \mapsto (I, J, I^c, J^c)$. Note that $**\omega = (-1)^{p+q}$ and $\overline{\partial}^* = -*\overline{\partial}*$. We prove the second identity: indeed for $\omega \in \mathcal{E}^{p,q-1}(M)$, $\eta \in \mathcal{E}^{n-p,n-q}(M)$ by the usual differentiation rule and the fact that $\overline{\partial} = d$ on $(n, n-1)$ forms

$$(\overline{\partial}\omega, \eta) = \int_M \overline{\partial}\omega \wedge *\eta \, vol_M = (-1)^{p+q-1} \int_M \omega \wedge \overline{\partial}(*\eta) vol_M + \int_M d(\omega \wedge \eta) vol_M;$$

by Stokes Theorem the last term is 0 thus

$$(\overline{\partial}\omega, \eta) = -\int_M \omega \wedge *(*\overline{\partial} * \eta) vol_M$$

so $(\overline{\partial}\omega, \eta) = (\omega, - * \overline{\partial} * \eta)$ and we found the adjoint. It is also remarkable the relation

$$\omega \wedge *\eta = (\omega, \eta) vol_M = \overline{(\eta, \omega)} vol_M = \overline{\eta \wedge *\omega} = \pm \overline{*\omega \wedge \eta} \tag{1.7}$$

for all ω, η as above.

An easy computation shows that this new operatorial definition of the Laplacian, on functions, coincides up to a constant with the usual Laplacian of a function of complex variables, that is, the Laplacian of its real and imaginary parts, which by the equations 1.2 and 1.3 is easily seen to be $\sum_j \frac{1}{4} \frac{\partial^2 f}{\partial z_j \partial \overline{z}_j}$. Indeed if $f \in C^\infty(M)$, call $dz = dz_1 \wedge \ldots \wedge dz_n$ the differentials with respect to local coordinates $z_1, \ldots, z_n$ and σ_j the permutation that puts $d\overline{z}_j$ in the $(n+1)$-th place. Using the relations $\partial / \partial \overline{z}_j (\overline{\partial f / \partial \overline{z}_j}) = \partial^2 \overline{f} / \partial z_j \partial \overline{z}_j$ and $\overline{\partial^2 \overline{f} / \partial z_j \partial \overline{z}_j} = \partial^2 f / \partial z_j \partial \overline{z}_j$

$$\begin{aligned}
\overline{\partial}^* \overline{\partial}(f) &= - * \overline{\partial} * \left(\sum_j \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \right) \\
&= * \overline{\partial} \left(2^{1-n} \sum_j - sgn(\sigma_j) \overline{\frac{\partial f}{\partial \overline{z}_j}} dz \wedge d\overline{z}_{j^c} \right) \\
&= * \left(2^{n-1} \sum_j - \frac{\partial^2 \overline{f}}{\partial z_j \partial \overline{z}_j} \right) dz \wedge d\overline{z} \\
&= 2 \sum_j - \frac{\partial^2 f}{\partial z_j \partial \overline{z}_j}.
\end{aligned}$$

As for the functions we therefore define a harmonic (p, q) form to be a form such that $\Delta \omega = 0$. Hence for a distribution T we can define a Laplacian $\Delta : \mathcal{D}^n \to \mathcal{D}^n$ by $\Delta T(f) = T(\Delta f)$ or equivalently

$$\Delta T(f) = -T \left(\sum_j \frac{\partial^2 f}{\partial z_j \partial \overline{z}_j} \right),$$

neglecting a constant. This definition equally applies if M is not compact. It is an obvious but crucial remark that every (p, q) cocycle is harmonic; to say it all what is true is that the space of harmonic (p, q) forms on M is isomorphic to $H^{p,q}_{\overline{\partial}}(M)$, but this is part of Hodge Theorem and does not come into the subject.

1.6 Fine Sheaves

This section will investigate some remarkable properties of fine sheaves, in particular we will show that the cohomology of a sheaf can be calculated as the cohomology of the global sections of its fine resolutions. The sheaves $\mathcal{E}^{p,q}$, $\mathcal{D}^{p,q}$ analyzed until here are fine sheaves the reason being that they are sheaves of modules over the fine sheaf C^∞. In this section a sheaf takes values in the category of vector spaces over $\mathbb{C}$.

Since we are dealing with a paracompact space cohomology is always meant to be Čech's. For a locally finite covering $\mathcal{U}$ of any paracompact topological space X the group of Čech q-cochains with coefficients in $\mathcal{U}$ relative to $\mathcal{F}$ is denoted $C^q(\mathcal{U},\mathcal{F})$. The symbol $H^q(\mathcal{U},\mathcal{F})$ stands for cohomology groups of $\mathcal{F}$ on X with coefficients in the covering $\mathcal{U}$ and $H^q(X,\mathcal{F})$ is the cohomology group of $\mathcal{F}$ i.e. the direct limit of the groups above over all the open coverings of X. The coboundary operator is denoted δ.

As we will see later on, the theorem duality between forms and currents involves compactly supported sections, so it is important here to treat the general case of cohomology in an arbitrary *paracompactifying family of supports* ϕ. We recall that this means that ϕ is a collection of closed subsets of our manifold M such that

1. For any $C \in \phi$, if $C' \subset C$ is closed then $C' \in \phi$;
2. ϕ is closed under arbitrary union of elements ;
3. Every element of ϕ has a closed neighbourhood belonging to ϕ
4. Every element of ϕ is paracompact.

Given a sheaf $\mathcal{F}$ over X and such a family ϕ we may consider for $U \subset X$ the subgroup $\mathcal{F}_\phi(U) \subset \mathcal{F}(U)$ of sections s on U such that $\mathrm{Supp}(s) = \{z \in X | s(z) \neq 0\} \subset C \in \phi$. Call Γ_ϕ the functor associating $\mathcal{F}_\phi(U)$ with each open set U: it is a subsheaf of $\mathcal{F}$. For an open locally finite covering $\mathcal{U}$ of X we may form Čech cochains with supports in ϕ by considering the compactly supported cochains subgroup $C^q_\phi(\mathcal{U},\mathcal{F}) = \prod_{i_o \dots i_q} \Gamma_\phi(U_{i_0} \cap \dots \cap U_{i_q}, \mathcal{F})$ of $C^q(\mathcal{U},\mathcal{F})$.

The conditions above assure that the coboundary δ sends q-cochains supported in ϕ to $q+1$-cochains supported in ϕ: by taking the direct limit over a cofinal set of locally finite coverings $\mathcal{U}$ of M one obtains the Čech q-th cohomology group with supports in ϕ, denoted $H^q_\phi(X,\mathcal{F})$.

The two most important cases, and the only ones that will occur in this thesis, are when ϕ is either the class of all closed subsets of X, in which case cochains with supports in ϕ are just ordinary Čech cochains, or that of all compact subset of X. In this latter circumstance we will replace the subscript ϕ with $*$.

For any paracompactifying family of supports ϕ, the following basic results hold for the functor $H^q_\phi(X,\bullet)$:

- To a short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of sheaves on X, there is an associated long exact sequence of cohomology groups

$$\dots \to H^q_\phi(X,\mathcal{F}) \to H^q_\phi(X,\mathcal{G}) \to H^q_\phi(X,\mathcal{H}) \to H^{q+1}_\phi(X,\mathcal{F}) \to \dots$$

- There are functorial isomorphisms

$$\Gamma_\phi(X, \bullet) \cong H^0_\phi(X, \bullet).$$

Now we take a closer look at the *fine* sheaves. In general when one is given a collection of sections s_i of a sheaf on a paracompact space X over different open sets U_i of X there is an intrinsic way to sum all these sections: declare $s = \sum_i s_i$ to be the section on $\bigcup_i U_i$ whose stalks are $s(x) = \sum_i s_i(x)$.

Definition 1.6.1. Let $\mathcal{F}$ be a sheaf over a paracompact topological space X. Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open covering of X. A *partition of unity* subordinated to the covering $\mathcal{U}$ is a family of sheaf morphisms $\eta_\alpha : \mathcal{F} \to \mathcal{F}$ such that (we denote by η_α^U the corresponding morphism on the open sets and η_α^x the one induced on the stalk $\mathcal{F}_x$)

1. $\sum_\alpha \eta_\alpha^U(f) = f$ for each $f \in \mathcal{F}(U)$ and each $U \subset M$ open
2. $\forall \alpha\ \eta_\alpha^x(\mathcal{F}_x) = 0$ for every $x \in X \setminus K_\alpha$ where K_α is a closed subset of U_α

A sheaf is called *fine* if it admits a partition of unity subordinated to *any* locally finite open covering $\mathcal{U}$. As we will see shortly this notion generalizes that of partition of unity for smooth functions. Note that the sum in 1 is well defined because $\mathcal{U}$ is locally finite.

It is easy to prove that the sheaves $\mathcal{E}^{p,q}$ and $\mathcal{D}^{p,q}$ are fine sheaves

Proposition 1.6.2. *Let C^∞_M be the sheaf of differentiable functions over a (differentiable or complex analytic) manifold M. Then any sheaf of C^∞_M-modules is fine.*

Proof. Let $\{\rho_\alpha\}$ be a C^∞_M partition of unity in the analytic sense, that is, a collection of smooth functions adding up to 1 subordinated to a locally finite covering $\mathcal{U} = \{U_\alpha\}$ of M, with compact supports $K_\alpha \subset U_\alpha$. For each α set a sheaf morphism by multiplication for ρ_α: $\rho_\alpha^U(f) = f\rho_\alpha|_U$ for any $U \subset M$. We have $\sum_\alpha \rho_\alpha^U(f) = \sum_\alpha f\rho_\alpha|_U = f$.

To show 2 take a neighbourhood U of $x \in X \setminus K_\alpha$ such that $U \cap K_\alpha = \emptyset$; since $\rho_\alpha \equiv 0$ on U we have $\rho_\alpha^U \equiv 0$ so that in particular 2 holds. □

Another property of fine sheaves is that their higher cohomology groups vanish.

Proposition 1.6.3. *If $\mathcal{F}$ is a fine sheaf over a paracompact space X and ϕ a paracompactifying family of supports then $H^q_\phi(X, \mathcal{F}) = 0$ for $q > 0$.*

Proof. Fix a partition of unity $\{\eta_\alpha\}$ subordinated to a locally finite covering $\mathcal{U} = \{U_\alpha\}$ of X. We just have to show that for every q-cocycle f in $C^q_\phi(\mathcal{U}, \mathcal{F})$, $q > 0$, there exists $g \in C^{q-1}_\phi(\mathcal{U}, \mathcal{F})$ such that $\delta g = f$ and the result follows by passing to the direct limit.

Fix a certain α, choose such a cocycle f, and let $\sigma = (i_0 \dots i_{q-1})$. Consider the section $f_{\sigma\alpha}$ on with compact support on $U_\sigma \cap U_\alpha$, restrict it to U_α and apply η_α; $\eta_\alpha^{U_\alpha \cap U_\sigma}(f_{\sigma\alpha}) \in \mathcal{F}(U_\alpha \cap U_\sigma)$ by property 2 of fine sheaves has still support in $U_\alpha \cap U_\sigma$ and vanishes outside it. Thus we can define a new section on the whole U_σ by extending it to 0. Set

$$g_\sigma^\alpha = E_\sigma \eta_\alpha^{U_\alpha \cap U_\sigma}(f_{\sigma\alpha})$$

where E_σ denotes the 0 extension on $\mathcal{F}(U_\sigma)$; g_σ^α has support contained in K_σ so the $(q-1)$ cochain $g^\alpha := (g_\sigma^\alpha)_\sigma$ has supports in ϕ. Letting $\underline{\sigma} = (i_0 \dots i_q)$ and $\sigma_j = (i_0 \dots i_{j-1} i_{j+1} \dots i_q)$ we compute the coboundary of g^α:

$$(\delta g^\alpha)_{\underline{\sigma}} = \sum_{j=0}^{q} (-1)^j E_{\sigma_j} \eta_\alpha^{U_\alpha \cap U_{\sigma_j}} (f_{\sigma_j \alpha})|_{\underline{\sigma}} = \eta_\alpha^{U_\alpha \cap U_\sigma}(f_{\sigma\alpha}) - E_\sigma \eta_\alpha^{U_\alpha \cap U_\sigma}((\delta f)_{\sigma\alpha}). \tag{1.8}$$

Being f a cocycle the second summand is 0 and the sum $g = \sum_\alpha g^\alpha$ is well defined because $\mathcal{U}$ locally finite. Moreover $\mathrm{Supp} g \subset \bigcup_\alpha \mathrm{Supp} g^\alpha \in \phi$. By 1.8 and in view of the properties of η_α stalkwise we have, for all $x \in U_\sigma$

$$\begin{aligned}(\delta g)_\sigma(x) &= \sum_\alpha (\delta g^\alpha)_\sigma(x) = \sum_\alpha \eta_\alpha^{U_\alpha \cap U_\sigma}(f_{\sigma\alpha})(x) = \sum_\alpha \eta_\alpha^x(f_{\sigma\alpha}(x)) = \\ &= \sum_\alpha \eta_\alpha^x(f_\sigma(x)) = f_\sigma(x)\end{aligned}$$

so that $\delta g = f$. □

The next theorem links cohomology of a sheaf $\mathcal{F}$ to that of global sections of a fine resolution. A fine *resolution* of $\mathcal{F}$ is an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G}_0 \to \mathcal{G}_1 \to \dots$$

where the $\mathcal{G}_i$s are all fine sheaves.

Theorem 1.6.4. *Let $\mathcal{F}$ be a sheaf on a paracompact topological space X and*

$$0 \to \mathcal{F} \to \mathcal{G}_0 \xrightarrow{d_0} \mathcal{G}_1 \xrightarrow{d_1} \mathcal{G}_2 \xrightarrow{d_2} \dots$$

a fine resolution $\mathcal{G}^*$ of $\mathcal{F}$. Let ϕ be a paracompactifying family of supports for X and $H^q(\Gamma_\phi(\mathcal{G}^*))$ be the cohomology of the induced complex

$$0 \to \Gamma_\phi(X, \mathcal{G}_0) \to \Gamma_\phi(X, \mathcal{G}_1) \to \Gamma_\phi(X, \mathcal{G}_2) \to \dots$$

of global sections with coefficients in ϕ. Then $H^q_\phi(X, \mathcal{F}) \cong H^q(\Gamma_\phi(\mathcal{G}^*))$.

Proof. Let $d_q^* : \Gamma_\phi(X, \mathcal{G}_q) \to \Gamma(X, \mathcal{G}_{q+1})$ be the induced homomorphisms of d_q on global sections supported on ϕ; we have to show that for any $q > 0$ it is $H_\phi^q(X, \mathcal{F}) \cong ker d_q^* / d_{q-1}^*(\mathcal{G}_{q-1}(X))$.

Denoting with K_i the kernel of the sheaf morphism d_i we can split $\mathcal{F}^*$ into the short exact sequences

$$0 \to \mathcal{F} \to \mathcal{G}_0 \xrightarrow{d_0} K_1 \to 0$$

$$0 \to K_i \to \mathcal{G}_i \xrightarrow{d_i} K_{i+1} \to 0, \quad i \geq 1$$

indeed $d_i : G_i \to K_i$ is surjective by exactness in G_{i+1} and $K_i \to G_i$ is the inclusion so that the sequence is exact also in G_i. For $q = 0$ the statement comes from the left exactness of $\Gamma_\phi(X, \bullet) \cong H_\phi^0(X, \bullet)$, that is, taking the global sections of the first exact sequence, the sequence

$$0 \to \Gamma_\phi(X, \mathcal{F}) \to \Gamma_\phi(X, \mathcal{G}_0) \xrightarrow{d_0} \Gamma_\phi(X, K_1)$$

is exact so $H_\phi^0(X, \mathcal{F}) \cong \Gamma_\phi(X, \mathcal{F}) \cong ker d_0 = H^0(\Gamma_\phi(\mathcal{G}^*))$.

Consider the long exact cohomology sequence obtained by the first sequence

$$\ldots \to H_\phi^{q-1}(X, \mathcal{G}_0) \to H_\phi^{q-1}(X, K_1) \to H_\phi^q(X, \mathcal{F}) \to H_\phi^q(X, \mathcal{G}_0) \to \ldots$$

For $q = 1$ being $\mathcal{G}_0$ fine the rightmost term vanishes and, together with the use of the isomorphims $H_\phi^0(X, \mathcal{C}) \cong \Gamma_\phi(X, \mathcal{C})$, where $\mathcal{C} = \mathcal{G}_0, K_1$, this leaves

$$\ldots \to \Gamma_\phi(X, \mathcal{G}_0) \xrightarrow{d_0^*} \Gamma_\phi(X, K_1) \xrightarrow{\gamma} H^1(X, \mathcal{F}) \to 0$$

So by exactness and the fundamental theorem

$$H_\phi^1(X, \mathcal{F}) = \Gamma_\phi(X, K_1)/ker\gamma = \Gamma_\phi(X, K_1)/Im d_0^* = H^1(\Gamma_\phi(\mathcal{G}^*)).$$

If $q > 1$ also the rightmost term cancels, yielding

$$H_\phi^{q-1}(X, K_1) \cong H_\phi^q(X, \mathcal{F}). \tag{1.9}$$

Let us consider now the second sequence; it reads

$$\ldots H_\phi^{q-2}(X, \mathcal{G}_i) \to H_\phi^{q-2}(X, K_{i+1}) \to H_\phi^{q-1}(X, K_i) \to H_\phi^{q-1}(X, \mathcal{G}_i) \ldots$$

If $q = 2$ the rightmost term again cancels so that this time we get

$$\ldots \Gamma_\phi(X, \mathcal{G}_i) \xrightarrow{d_i^*} \Gamma_\phi(X, K_{i+1}) \to H_\phi^1(X, K_i) \to 0, \ \forall i \geq 1$$

which leads to $H_\phi^1(X, K_i) \cong ker d_{i+1}^* / Im d_i^* \cong H^i(\Gamma_\phi(\mathcal{G}^*))$. In particular for $i = 1$ from 1.9 we obtain $H_\phi^2(X, \mathcal{F}) \cong H^2(\Gamma_\phi(\mathcal{G}^*))$; if $q > 2$ and $i > 1$ it is $H_\phi^{q-2}(X, K_i) \cong H_\phi^{q-1}(X, K_{i-1})$ since $H_\phi^{q-2}(X, \mathcal{G}_i) = H_\phi^{q-1}(X, \mathcal{G}_i) = 0$. Iterating this isomorphism for increasing q and using again 1.9 we have $H_\phi^1(X, K_q) \cong H_\phi^{q-1}(X, K_1) \cong H_\phi^q(X, \mathcal{F})$; but $H^1(X, K_q)$ was seen to be isomorphic to $H^q(\Gamma_\phi(\mathcal{G}^*))$ so the theorem is proved for any q. □

Chapter 2

The Duality Theorem

We start out from the complexes given in 1.4 and 1.6. The strategy of the proof is the following: first we show that they provide different fine resolutions of the sheaf of holomorphic p-forms then we generalize this to forms and currents with coefficients in an arbitrary locally sheaf $\mathcal{V}$ (section 2.1). Besides, if we tensor 1.4 with such a sheaf and 1.6 with its dual $\mathcal{V}^*$ we may prove that there is indeed a (topological) duality between the two induced complexes of global sections. Finally this duality will be extended to cohomology groups to obtain the Theorem in its full generality (section 2.2)

In section 2.3 some classical applications as the Riemann-Roch Theorem will be treated, along with some examples.

2.1 The Sheaf of Holomorphic Forms and its Resolutions

Let us rigourosly define the sheaf of holomorphic forms, already introduced in chapter 1.

Definition 2.1.1. Let M be a n-dimensional complex analytic manifold. The *sheaf of holomorphic p-forms on* M is the sheaf Ω^p of holomorphic sections of the vector bundle $\Lambda^{p,0}(M)$. The canonical sheaf of M defined in 1.1.4 is then just Ω^n

Clearly from the definition follows that Ω^p is a subsheaf of $\mathcal{E}^p$. Furthermore for each $U \subset M$ the space $\Omega^p(U)$ of holomorphic p-forms on U is the subspace of $\Omega^p(U)$ of elements whose local expressions $\sum_I f_I dz_I$ are such that f_I is holomorphic in U for every I. Our first goal is the proof of the following theorem:

Theorem 2.1.2. *Let be Ω^p be the sheaf of holomorphic differentials on a complex analytic manifold M of dimension n. The cochain complexes 1.4 and 1.6 extend to complexes of sheaves*

$$0 \to \Omega^p \to \mathcal{E}^{p,0} \to \mathcal{E}^{p,1} \ldots \to \mathcal{E}^{p,n} \to 0. \tag{2.1}$$

and

$$0 \to \Omega^p \to \mathcal{D}^{p,0} \to \mathcal{D}^{p,1} \ldots \to \mathcal{D}^{p,n} \to 0. \tag{2.2}$$

that are furthermore exact, thus representing two different fine resolutions of Ω^p.

The main point is to prove that locally every $\overline{\partial}$-closed (p, q) form or (p, q) current is $\overline{\partial}$-exact, so essentially we must establish versions of the Poincaré Lemma for both forms and currents. This matter is deeply analytic. We shall prove first all the needed machinery of complex calculus and harmonic theory on manifolds and then proceed with the proof. We first deal with forms, starting with a mild, almost tautological, proposition.

Proposition 2.1.3. *The kernel of* $\overline{\partial} : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1}$ *is* Ω^p.

Proof. A section $\omega = \sum_I f_I dz_I$ on $U \subset M$ is such that $\overline{\partial}(\omega) = 0$ if and only if f_I is holomorphic in every argument for each I since $\partial f_I / \overline{\partial} z_j = 0 \ \forall j$ is an equivalent formulation of the Cauchy-Riemann conditions. To support this claim write a smooth function f as a sum of real and imaginary parts of its local coordinates: $f(z_1, \ldots, z_n) = u(x_1 \ldots x_n, y_1, \ldots, y_n) + iv(x_1, \ldots, x_n, y_1, \ldots, y_n)$ for $z_j = x_j + iy_j$, $j = 1, \ldots, n$. Then using equations 1.2 and 1.3 we have

$$0 = \frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} u - \frac{\partial}{\partial y_j} v + i \left(\frac{\partial}{\partial y_j} u + \frac{\partial}{\partial z_j} v \right) \right)$$

if and only if

$$\frac{\partial}{\partial x_j} u = \frac{\partial}{\partial y_j} v \, , \ \frac{\partial}{\partial y_j} u = -\frac{\partial}{\partial x_j} v.$$

□

We will now recall the generalized Cauchy formula for a C^∞ function of complex variable, that plays just the same role of the fundamental theorem of calculus in the ordinary Poincaré Lemma for differential forms.

Theorem 2.1.4 (Generalized Cauchy Integral Formula). *Let* $f \in C^\infty(U)$ *for an open set* $U \subset \mathbb{C}^n$ *containing an open disc* D *with boundary* γ *whose closure is contained in* U*. Then for every* $z \in D$

$$2\pi i f(z) = \int_\gamma f(w) \frac{dw}{w - z} - \int_D \frac{\partial f(w)}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z} \tag{2.3}$$

$$2\pi i f(z) = \int_\gamma f(w) \frac{d\overline{w}}{\overline{w} - \overline{z}} + \int_D \frac{\partial f(w)}{\partial w} \frac{dw \wedge d\overline{w}}{\overline{w} - \overline{z}} \tag{2.4}$$

Proof. Let $z \in D$ and D_r a disc of radius $r > 0$ with closure contained in D with boundary γ_r. By Stokes Theorem

$$\int_\gamma f(w) \frac{dw}{w - z} - \int_{\gamma_r} f(w) \frac{dw}{w - z} = \int_{D \backslash D_r} d \left(f(w) \frac{dw}{w - z} \right) = \int_{D \backslash D_r} \frac{\partial f(w)}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w - z} \tag{2.5}$$

Now if $r \to 0$ the last term converges to the integral over the whole D because $\partial f(w)/\partial \overline{w}$ is bounded on D and $dz \wedge d\overline{z}/(w-z)$ is integrable in a neighbourhood of the singularity z; on the other hand the integral over γ_r, parametrizing $w = z + re^{it}$, $0 \leq t < 2\pi$, writes as

$$\int_{\gamma_r} f(w) \frac{dw}{w-z} = \int_0^{2\pi} f(z + re^{it}) i dt.$$

For fixed z and r the functions of $f(z + re^i t)$ of t are dominated by $\sup_D f(w)$ and converge to $f(z)$ for $r \to 0$, so by the dominated convergence theorem we may pass the limit under the integral sign obtaining

$$\lim_{r \to 0} \int_{\gamma_r} f(w) \frac{dw}{w-z} = \int_0^{2\pi} f(z) i dt = 2\pi i f(z).$$

Hence 2.5 becomes

$$\int_{\gamma} f(w) \frac{dw}{w-z} - 2\pi i f(z) = \int_D \frac{\partial f(w)}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z}$$

which is 2.3. 2.4 follows from this by taking the complex conjugate. □

By means of this equation we can prove the next lemma:

Lemma 2.1.5. *Let f, U and D as in Theorem 2.1.4. There exists a C^∞ function g such that $\partial g/\partial \overline{z} = f$ in D.*

Proof. Introduce

$$g(z) = \frac{1}{2\pi i} \int_D f(w) \frac{dw \wedge d\overline{w}}{w-z}; \tag{2.6}$$

this will be our primitive. For $z \in D$ choose a disc D_r centered at z with closure contained in D and call γ_r its boundary. In D a branch of the complex logarithm is defined, so by Stokes Theorem

$$\int_{\gamma} f(w) \log |w-z|^2 d\overline{w} - \int_{\gamma_r} f(w) \log |w-z|^2 d\overline{w} = \int_{D \setminus D_r} d(f(w) \log |w-z|^2 d\overline{w}); \tag{2.7}$$

but since $\log |w-z|^2 = \log(w-z) + \log(\overline{w} - \overline{z})$ differentiating the integrand yields

$$d(f(w) \log |w-z|^2) = \frac{\partial f(w)}{\partial w} \log |w-z|^2 dw + f(w) \frac{dw}{w-z} + \frac{\partial f(w)}{\partial \overline{w}} \log |w-z|^2 d\overline{w} + f(w) \frac{d\overline{w}}{\overline{w} - \overline{z}}.$$

By substituting in 2.7 the last two terms vanish by wedging against $d\overline{w}$, and what is left is

$$\int_{\gamma} f(w) \log |w-z|^2 d\overline{w} - \int_{\gamma_r} f(w) \log |w-z|^2 d\overline{w} =$$

$$= \int_{D \setminus D_r} \frac{\partial f(w)}{\partial w} \log |w-z|^2 dw \wedge d\overline{w} + \int_{D \setminus D_r} f(w) \frac{dw \wedge d\overline{w}}{w-z}. \tag{2.8}$$

Now let $w = z + re^{it}$, $0 \le t < 2\pi$, the standard parametrization of the disc, $M = \sup_{z\in D} |f(z)|$ and let $r \to 0$; it is

$$\begin{aligned}\lim_{r\to 0}\left|\int_{\gamma_r} f(w)\log|w-z|^2 d\overline{w}\right| &= \lim_{r\to 0}\left|2i\int_0^{2\pi} f(z+re^{it})\log(r)re^{-it}dt\right| \le \\ &\le \lim_{r\to 0} 4\pi M r\log r = 0\end{aligned}$$

so that in the limit equation 2.8 becomes

$$\int_\gamma f(w)\log|w-z|^2 d\overline{w} = \int_D \frac{\partial f(w)}{\partial w}\log|w-z|^2 dw\wedge d\overline{w} + 2\pi i g(z)$$

by using 2.6. Differentiation under integral by $\partial/\partial\overline{z}$ is allowed because after deriving the left side is still integrable, so we finally have

$$-\int_\gamma f(w)\frac{d\overline{w}}{\overline{w}-\overline{z}} = -\int_D \frac{\partial f(w)}{\partial w}\frac{dw\wedge d\overline{w}}{\overline{w}-\overline{z}} + 2\pi i\frac{\partial g(z)}{\partial\overline{z}}$$

and thus the differential relation follows form 2.4; of course, again by 2.4 the function $\partial g(z)/\partial\overline{z}$ is C^∞ so that also $g(z)$ is. □

Remark. Think for a moment that U is itself a disc. This lemma assures that f has a primitive in any smaller disc $D' \subset U$ but does not guarantee alone that $\partial g/\partial\overline{z} = f$ on $\overline{U}$. Indeed *on discs* the above lemma can be improved to obtain a function verifying this, but only due to the special geometry of U. We do not need this version tough. Note also that such a function g retains differentiability (or analiticity) with respect to any other variables f may involve, and this is simply seen by deriving under integral sign.

This result is basically all that underlies the following version of the Poincaré Lemma.

Proposition 2.1.6 ($\overline{\partial}$ Poincaré Lemma). *Let D be a disc $D = \{z \in \mathbb{C}^n, |z_j - z_0| < r, j = 1,\dots,n\}$ of radius r centered at z_0. For $q > 0$ let ω be a $\overline{\partial}$-closed (p,q) form on D; then for any smaller disc $D' \subset D$ there exists a $(p, q-1)$ form η such that $\overline{\partial}\eta = \omega$ in D'.*

Proof. Choose a (p,q) cocycle ω as in the hypotheses and write it on D' as

$$\omega = \sum_{\substack{i_1\dots i_p\\ j_1\dots j_q}} f_{i_i\dots i_p,j_1\dots j_q} dz_{i_1}\wedge\ldots\wedge dz_{i_p}\wedge d\overline{z}_{j_1}\wedge\ldots\wedge d\overline{z}_{j_q}.$$

We proceed formally by induction on the last integer m appearing in the complex conjugate coordinate representation of ω.

If $m = 0$ then $\omega = 0$ is in $\mathcal{E}^{p,0}$ and there is nothing to prove. If $m > 0$ and $\omega \ne 0$ then m coincides with j_q. Therefore we can write ω as $\omega = (\alpha\wedge d\overline{z}_m) + \beta$ for appropriate forms α and

β whose anti-holomorphic part involve only differentials $d\bar{z}_i$ for $i \in \{1, \dots, m-1\}$. Applying $\overline{\partial}$ we obtain

$$0 = \overline{\partial}\omega = (\overline{\partial}\alpha \wedge d\bar{z}_m) + \overline{\partial}\beta.$$

Call $g_{I,J}$ the coefficients of α. Since α and β do not contain differentials $d\bar{z}_k$ for $m < k < n$, the only way in which after differentiation terms containing these expressions may be 0 is that $\partial f_{I,J}/\partial \bar{z}_k = 0$ for all multi-indices I and J of length p and $q-1$ and for all $k > m$. Thus each $f_{I,J}$ is a holomorphic function in the variables $z_{m+1}, \dots, z_n$, as well as C^∞ in $z_1, \dots, z_m$, in the respective domains $D_k = \{z_k : |z_k - z_0| < r'\}$. By lemma 2.1.5 for every I, J there exist differentiable functions $g_{I,J}$ such that $\partial g_{I,J}/\partial \bar{z}_m = f_{I,J}$ in any disc $D'_k \subset D_k$. Set

$$\gamma = \sum_{I,J} g_{I,J} dz_I \wedge d\bar{z}_J.$$

Recalling the remark following 2.1.5, for $k > m$ it is $\partial g_{I,J}/\overline{\partial} z_k = 0$, so that

$$\overline{\partial}\gamma = \sum_{I,J,k=0}^{n} \frac{\partial g_{I,J}}{\partial \bar{z}_k} dz_I \wedge d\bar{z}_J \wedge d\bar{z}_k = \sum_{I,J,k<m} \frac{\partial g_{I,J}}{\partial \bar{z}_k} dz_I \wedge d\bar{z}_J \wedge d\bar{z}_k +$$
$$+ \sum_{I,J} f_{I,J} dz_I \wedge d\bar{z}_J \wedge d\bar{z}_m = \delta + (\alpha \wedge d\bar{z}_m),$$

where in δ still only appear conjugate differentials up to $m-1$. The same holds for the form $\phi = \omega - \overline{\partial}\gamma = \beta - \delta$ which moreover satisfies $\overline{\partial}\phi = \overline{\partial}\omega - \overline{\partial}\,\overline{\partial}\gamma = 0$ so by the induction hypothesis a $(p, q-1)$ form ψ such that $\overline{\partial}\psi = \phi$ in D' exists, whence finally $\omega = \overline{\partial}(\gamma + \psi)$, and the proposition is proved. □

Again we remark that this does not prove that a global form η can be found such that $\overline{\partial}\eta = \omega$ on the whole D; in this case this turns out to be true, but, as before, only because we are dealing with discs. We can indeed extend the above proposition to any class of domains U in $\mathbb{C}^n$ but the general statement 'for any open set $U \subset \mathbb{C}^n$ and for all $\omega \in \mathcal{E}^{p,q}(U)$ there exists $\eta \in \mathcal{E}^{p,q-1}(U)$ such that $\overline{\partial}\eta = \omega$ on U', is false. An easy counterexample is the differential form $d\bar{z}/\bar{z}$ on $\mathbb{C} \setminus \{0\}$, whose primitive $\log \bar{z}$ is defined every disc but cannot be extended globally.

For what concerns currents the equivalent proposition will be derived with the help of harmonic theory on the complex n dimensional space, using the ideas and definitions roughly introduced in section 1.5. We begin with the proof of the classical *integral mean theorem* for harmonic functions on $\mathbb{C}^n$.

Proposition 2.1.7. *If $f \in C^\infty(\mathbb{C}^n)$ is a harmonic function then for every $z \in \mathbb{C}^n$ and $r > 0$*

$$f(z) = \int_{\partial D_r(z)} f(w)\sigma(w - z)$$

where σ is a never vanishing $(n, n-1)$ form, radially symmetric with respect to the Euclidean norm $\|\cdot\|$ of the underlying $2n$-dimensional real structure, and such that its integral over $D_r(z)$ is 1 for every $r > 0$.

Proof. Define

$$\tau = \sum_i^n (-1)^{i+1} \frac{\overline{z}_i}{\|z\|^{2n}} d\overline{z}_1 \wedge \ldots \wedge \hat{d\overline{z}_i} \wedge \ldots \wedge d\overline{z}_n \wedge dz_1 \wedge \ldots \wedge dz_n$$

Note that in terms of the Hodge operator defined in section 1.5, with respect to the canonical volume form $dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n$ of $\mathbb{C}^n$, τ equals $*\left(\frac{\sum_i z_i d\overline{z}_i}{\|z\|^{2n}}\right)$. Observe moreover that τ is closed:

$$\begin{aligned}
\overline{\partial}\tau(z) &= \left[\sum_i (-1)^{i+1} \left(\sum_j \frac{\overline{\partial}}{\partial \overline{z}_j} \frac{\overline{z}_i}{\| z \|^{2n}} d\overline{z}_j\right) \wedge d\overline{z}_1 \wedge \ldots \wedge \hat{d\overline{z}_i} \wedge \ldots \wedge d\overline{z}_n\right] \\
&= n \sum_i \frac{1}{\|z\|^{2n}} d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n - n \sum_i \frac{z_i \overline{z}_i}{\|z\|^{2(n+1)}} d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n \\
&= 0.
\end{aligned}$$

Now if $B_{r,r'} = D_r(z) \setminus D_{r'}(z)$ is the spherical shell between the discs of radiuses $r' < r$ by Stokes Theorem we see that τ has integral 0 over B so that

$$\int_{D_r(z)} \tau(w - z)$$

does not depend on r (its value is actually given by the 'residue' of τ in the singularity z). We can therefore adjust τ by multiplying it with a constant C_n in such a way that the above integral is 1 on every disc, and moreover it is not reductive to assume $z = 0$. In particular if $r = 1$ the form $*\left(\sum_i w_i d\overline{w}_i\right)$ on $D_1(0)$ is just the volume form $d\theta$ of the underlying $2n - 1$ real sphere, so we see that C_n must be the inverse of the volume of S^{2n-1}. We set $\sigma = C_n\tau$. Take the $(n, n-1)$ form $\eta = f\sigma$; remember that on such forms $\overline{\partial} = d$. Since $\overline{\partial}\sigma = 0$ the differentiation rule together with 1.7 gives

$$\begin{aligned}
\overline{\partial}\eta &= \overline{\partial} f \wedge \sigma \\
&= \overline{\partial} f \wedge * C_n \left(\sum_i \frac{w_i}{\|w\|^{2n}} d\overline{w}_i\right) \\
&= \overline{\pm * \overline{\partial} f \wedge C_n \left(\sum_i \frac{w_i}{\|w\|^{2n}} d\overline{w}_i\right)} \\
&= \pm * \overline{\overline{\partial} f} \wedge C_n \left(\sum_i \frac{\overline{w}_i}{\|w\|^{2n}} d\overline{w}_i\right)
\end{aligned}$$

By applying Stokes Theorem on the shell $B_{r,r'}$ we have

$$\int_{\partial D_r(0)} \eta(w) - \int_{\partial D_{r'}(0)} \eta(w) = \pm C_n \int_{B_{r,r'}} *\overline{\overline{\partial} f(w)} \wedge \sum_i \frac{\overline{w}_i}{\|w\|^{2n}} d\overline{w}_i = \pm C_n \int_{B_{r,r'}} *\overline{\overline{\partial} f(w)} \wedge \overline{\overline{\partial} \gamma(w)} \tag{2.9}$$

where γ is the function

$$\gamma = \begin{cases} \log \|w\|^2 & \text{if } n = 1 \\ \frac{1}{1-n} \frac{1}{\|w\|^{2n-2}} & \text{if } n > 1 \end{cases}$$

Now f is harmonic so $\overline{\partial} * \overline{\partial} f = \pm \Delta f dw_1 \wedge \ldots \wedge d\overline{w}_n = 0$ therefore again by the Stokes Theorem and the differentiation rule (and the fact that $\gamma = \overline{\gamma}$), the right side of 2.9 is equal to

$$C_n \left[\int_{\|z\|=r} *\overline{\overline{\partial} f} \gamma - \int_{\|z\|=r'} *\overline{\overline{\partial} f} \gamma \right] = C_n \left[c_1 \int_{\|z\|=r} *\overline{\overline{\partial} f} - c_2 \int_{\|z\|=r'} *\overline{\overline{\partial} f} \right]$$

for certain constants c_1, c_2, being γ radially symmetric. By using a third time Stokes the two last integrals cancel, again by virtue of the harmonicity of f. Hence the difference on the left side of 2.9 is 0; we have then for any $r' < r$

$$\int_{\partial D_r(0)} \eta = \int_{\partial D_{r'}(0)} \eta \to f(0)$$

as $r' \to 0$; indeed the integral

$$\int_{\partial D_{r'}(0)} f\sigma$$

is always between $\inf_{D_{r'}(0)} f$ and $\sup_{D_{r'}(0)} f$ being 1 the total integral of σ, regardless of r'. $\square$

As a consequence of this result we have a sort of 'regularity property' for harmonic distributions:

Proposition 2.1.8. *Let T be a distribution on $\mathbb{C}^n$ such that $\Delta T = 0$. Then T is regular, that is, exists $h \in C^\infty(\mathbb{C}^n)$ such that $T = T_h$.*

Proof. By proposition 2.1.7 for every $f \in C^\infty(\mathbb{C}^n)$ harmonic for every $\delta > 0$ its regularization f_δ by a function χ_δ as in section 1.5 satisfies $f_\delta = f$. In order to prove this claim we must use the properties of σ. A number $w \in \mathbb{C}^n$ in polar coordinates is in the form $w = r\theta$ where $r > 0$, and θ parametrizes a point of the underlying $2n-1$ real unitary sphere S^{2n-1}. It is $\sigma(r\theta) = C_n d\theta$ according to the remarks in the previous proposition. If $dw = dw_1 \wedge \ldots \wedge dw_n \wedge d\overline{w}_1 \wedge \ldots \wedge d\overline{w}_n$, by choosing χ_δ supported on $D_\delta(z)$ with total integral C_n, C_n as in proposition 2.1.7, we have, $\forall z \in \mathbb{C}^n$

$$f_\delta(z) = \int_{D_\delta(0)} f(z+w)\chi_\delta(w)dw = \int_{D_\delta(0)} f(z+r\theta)\chi_\delta(r)drd\theta$$

having used the substitution $w \mapsto z + w$ and the fact that $\chi_\delta(r\theta) = \chi_\delta(r)$. We can therefore integrate first with respect to θ and then use the mean property of f

$$\begin{aligned} f_\delta(z) &= \int_{D_\delta(0)} \chi_\delta(r) \left(\int_{S^{2n-1}} f(z + r\theta) d\theta \right) dr = \\ &= \int_{D_\delta(0)} \chi_\delta(r) \left(\frac{1}{C_n} \int_{S^{2n-1}} f(z + r\theta)\sigma(r\theta) \right) dr = \\ &= \frac{1}{C_n} \int_{D_\delta(0)} \chi_\delta(r) f(z) dr = f(z). \end{aligned}$$

Let T be a distribution such that $\Delta T = 0$; for every $\epsilon > 0$ let χ_ϵ as in section 1.5. Then the function ΔT_ϵ is such that $\Delta T_\epsilon = -T(\Delta \chi_\epsilon(z - w)) = 0$. By property 3 of regularization we have then that $(\Delta T)_\epsilon = \Delta T_\epsilon = 0$; if we consider $(\Delta T)_\epsilon$ as the distribution given by integrating with the smooth function $(\Delta T)_\epsilon$, it is, for all $g \in C^\infty_*(\mathbb{C}^n)$

$$0 = (\Delta T)_\epsilon(g) = \int_{z \in \mathbb{C}^n} g(z) \Delta T(\chi_\epsilon(z - w)) dz,$$

which implies $\Delta T(\chi_\epsilon(z - w)) = 0$. So for each ϵ as a distribution T_ϵ acts like

$$T_\epsilon(g) = \int_{z \in \mathbb{C}^n} g(z) T(\chi_\epsilon(z - w)) dz,$$

that is, $T_\epsilon = T_{f^\epsilon}$ where $f^\epsilon(z)$ is the harmonic function $T(\chi_\epsilon(z - w))$. Now for any $\delta > 0$ the convolution product $(f^\epsilon)_\delta$ equals f^ϵ; by the formal use of properties $1 \div 3$ of section 1.5 it follows that for any $g \in C^\infty_*(\mathbb{C}^n)$

$$\begin{aligned} T(g) &= \lim_{\epsilon \to 0} T_\epsilon(g) = \lim_{\epsilon \to 0} T_{f^\epsilon}(g) = \lim_{\epsilon \to 0} T_{f^\epsilon_\delta}(g) = \\ &= \lim_{\epsilon \to 0} (T_{f^\epsilon})_\delta(g) = \lim_{\epsilon \to 0} T_{f^\epsilon}(g_\delta) = T(g_\delta) = T_\delta(g) \end{aligned}$$

so that T is the smooth distribution given by integration with the smooth function $h = T_\delta$. $\square$

This is a first step towards the exactness of 2.2 as reveals the following

Corollary 2.1.9. *If $U \subset \mathbb{C}^n$ is an open subset and T is a distribution on U such that $\overline{\partial} T = 0$, then there exists a holomorphic function f on U such that $T = T_f$.*

Proof. This is actually two corollaries in one: first we must prove that the regularity of harmonic distributions also holds on open subsets of $\mathbb{C}^n$ and then that it transfers to holomorphic functions. Let $U \subset \mathbb{C}^n$ be an open set $f \in C^\infty_*(U)$ an arbitrary harmonic function. Fix $\epsilon_0 > 0$. There exists a subset V of U such that for every $z \in V$ it is $D_\epsilon(z) \subset U$ for any $\epsilon < \epsilon_0$; by the proof of proposition 2.1.8 for any such ϵ it is $f_\epsilon(w) = f(w)$ for every $w \in D_\epsilon(z)$.

Let V_0 be the maximal set with this property. By using exactly the same formal passages of 2.1.8, this time taking χ_δ with $\delta < \epsilon_0$, we can find a function g_0 harmonic in V_0 such that

$T = T_{g_0}$ on V_0. By choosing a sequence $\epsilon_i \to 0$ we can iterate the argument and find a sequence (V_i, g_i) with V_i open subsets $V_i \subset V_{i+1}$ invading U and g_i harmonic on V_i such that $T = T_{g_i}$ on V_i and $g_i|_{V_{i-1}} = g_{i-1}$. Therefore for $z \in U$ the function $g(z) = g_i(z)$ if $z \in V_i$ is harmonic on U and such that $T = T_g$.

For what concerns the second claim $\overline{\partial}T = 0$ implies $\Delta T = 0$ as remarked in section 1.5; by the regularity of T does exist $f \in C^\infty(U)$ such that $T = T_f$; but then

$$0 = \overline{\partial}T = \overline{\partial}T_f = T_{\overline{\partial}f}$$

means $\overline{\partial}f = 0$, i.e. f is holomorphic in U. □

We must now find some way to generalize proposition 2.1.6; rather than starting out a new approach on the dual space it is easier to give a new proof of the case of forms which can be extended to currents. To do that we must introduce a homotopy operator between the morphisms of complexes of sheaves $id : \mathcal{E}^{p,*}_* \to \mathcal{E}^{p,*}$, and $0 : \mathcal{E}^{p,*}_* \to \mathcal{E}^{p,*}$, that is a collection of maps

$$K : \mathcal{E}^{p,q}_*(U) \to \mathcal{E}^{p,q-1}(U)$$

such that

$$id = K\overline{\partial} + \overline{\partial}K.$$

for an appropriate class of open subsets $U \subset \mathbb{C}^n$, namely regular domains. If this can be done then for a cocycle $\omega \in \mathcal{E}^{p,q}(U)$ choose a $C^\infty(U)$ bump ρ which is 1 on a slightly smaller relatively compact regular domain $V \subset U$. For $z \in V$ we have

$$\omega(z) = \rho\omega(z) = \overline{\partial}K(\rho\omega)(z) + K\overline{\partial}(\rho\omega)(z) = \overline{\partial}K(\rho\omega)(z)$$

which is proposition 2.1.6. Let us transpose this to currents. Suppose we are given the homotopy K; if T is compactly supported we define $K' : \mathcal{D}^{p,q}_*(U) \to \mathcal{D}^{p,q-1}(U)$ in the natural way: for $\omega \in \mathcal{E}^{n-p,n-q+1}_*(U)$ let K' be given by

$$K'T(\omega) = T(K\omega);$$

it is well defined even if $K\omega$ doesn't have compact support, because T does. K' is a chain homotopy since

$$\overline{\partial}K'T(\omega) + K'\overline{\partial}T(\omega) = K'T(\overline{\partial}\omega) + \overline{\partial}T(K\omega) = T(K\overline{\partial}\omega) + T(\overline{\partial}K\omega) = T(K\overline{\partial}\omega + \overline{\partial}K\omega) = T(\omega)$$

Again if ρ is a C^∞ partition of unity with value 1 in $V \subset U$ and T a cocycle in $\mathcal{D}^{p,q}(U)$

$$T(\omega) = \rho T(\omega) = \overline{\partial}K'\rho T(\omega) + K'\overline{\partial}\rho T(\omega) = \overline{\partial}K'\rho T(\omega) \tag{2.10}$$

for every $\omega \in \mathcal{E}^{n-p,n-q}_*(V)$, and this is the Poincaré Lemma for currents.

We must then only find the homotopy K. First some notations. I and J are ordered multi-indeces of length p and q and $dz_I, d\overline{z}_J$ are as usual differentials of wedge products relative to I and J of length p and q. Morover we set

$$dz = dz_1 \wedge \ldots \wedge dz_n, \quad d\overline{z} = d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n$$
$$dz[I] = dz_{I^c},\ dz[J] = dz_{J^c} \quad \text{(taken in } \{1,\ldots,p\} \text{ and } \{1,\ldots,q\})$$
$$\sigma(I) = sgn\left((1,\ldots,p) \mapsto (I, I^c)\right), \quad \sigma(J) = sgn\left((1,\ldots,q) \mapsto (J, J^c)\right)$$

Finally let c_n the volume of $D_1(0)$. We introduce the *Bochner-Martinelli* kernel, that is the bigraded form

$$k_{p,q}(w,z) = (-1)^{p(n-q-1)}\frac{1}{nC_n}\sum_{I,J}\sum_{k\notin J}\sigma(k,J)\sigma(I)\frac{(\overline{w}_k - \overline{z}_k)}{\| z-k \|^{2n}}d\overline{w}[k,J]\wedge dw[I]\wedge d\overline{z}_J\wedge dz_I; \quad (2.11)$$

it is a (p,q) form in the variables z_i and a $(n-p, n-q-1)$ form in the variables w_i. Then we set $K : \mathcal{E}^{p,q}_*(U) \to \mathcal{E}^{p,q-1}(U)$ to be

$$K\omega(z) = -\int_{w\in U}\omega(w)\wedge k_{p,q-1}(w,z)\,, \qquad \forall z \in U$$

for any $\omega \in \mathcal{E}^{p,q}_*(U)$. Clearly there is nothing wrong with this integral being a sum of wedge products having coefficients $f_k z_k/\|z-w\|^{2n}$, $k = 1,\ldots,n$ for smooth functions f_k with compact support, and these quotients are integrable on a neighbourhood of the singularities $z = w$ in $U \subset \mathbb{C}^n$. It is also remarkable that $\overline{\partial}_w k_{p,q}(w,z) = (-1)^{p+q}\overline{\partial}_z k_{p,q-1}(w,z)$. We proceed to prove that K is a homotopy: in its analytic form what we want to prove is

Proposition 2.1.10. *Let $U \subset \mathbb{C}^n$ a domain with smooth boundary and ω a (p,q) form with compact support contained in U. Then $\forall z \in U$*

$$-\omega(z) = \int_{w\in U}\overline{\partial}\omega(w)\wedge k_{p,q}(w,z) + \overline{\partial}\int_{w\in U}\omega(w)\wedge k_{p,q-1}(w,z).$$

We break the proposition into a sequence of lemmas.

Lemma 2.1.11. *Let $f \in C^\infty(U)$, with $U \subset \mathbb{C}^n$ open. For any disc $D_r(z) \subset U$ whose closure is contained in U it is*

$$\lim_{r\to 0}\int_{\partial D_r(z)} f(w)\frac{d\overline{w}[k]\wedge dw}{\|w-z\|^{2n}} = (-1)^{k+1}c_n\frac{\partial}{\partial \overline{z}_k}f(z),$$

for $k = 1,\ldots,q$, and

$$\lim_{r\to 0}\int_{\partial D_r(z)} f(w)\frac{d\overline{w}\wedge dw[k]}{\|w-z\|^{2n}} = (-1)^{n+k+1}c_n\frac{\partial}{\partial z_k}f(z).$$

for $k = 1,\ldots,p$.

Proof. This is just Stokes Theorem combined with the integral mean value theorem. On $(n, n-1)$ forms $\overline{\partial} = d$ so

$$\int_{\partial D_r(z)} f(w)\frac{d\overline{w} \wedge dw[k]}{\|w-z\|^{2n}} = \frac{1}{r^{2n}}\int_{D_r(z)} d(f(w)d\overline{w}[k] \wedge dw) =$$

$$= \frac{(-1)^{k+1}}{r^{2n}}\int_{D_r(z)} \frac{\partial}{\partial \overline{w}_k} f(w) d\overline{w} \wedge dw$$

$$\Rightarrow \frac{(-1)^{k+1}}{r^{2n}}\mathrm{vol}(D_r(z)) \inf_{D_r(z)} \frac{\partial}{\partial \overline{w}_k} f(z) \leq \int_{\partial D_r(z)} f(w)\frac{d\overline{w} \wedge dw[k]}{\|w-z\|^{2n}} \leq$$
$$\frac{(-1)^{k+1}}{r^{2n}}\mathrm{vol}(D_r(z)) \sup_{D_r(z)} \frac{\partial}{\partial \overline{w}_k} f(z)$$

and both sides of the inequality converge to the desired function of z as $r \to 0$ because $\mathrm{vol}(D_r(z)) = c_n r^{2n}$. The second limit is the same but this time there are n more changes of places to be made to bring dz_k in its correct place. □

Lemma 2.1.12. *Let ω be a (p, q) differential form with compact support on U and $D_r(z)$ a disc contained in the support of ω. Then*

$$\lim_{r \to 0} \int_{\partial D_r(z)} \omega(w) \wedge k_{p,q}(w, z) = \left(1 - \frac{q}{n}\right)\omega(z).$$

Proof. In this lemma and in the following ones we can suppose, without loss of generality, that

$$\omega(w) = f(w) dw_H \wedge d\overline{w}_K$$

for arbitrary multi-indices H and K of length p and q. Then

$$\int_{\partial D_r(z)} \omega(w) \wedge k_{p,q}(w, z) =$$
$$= \frac{1}{r^{2n} n C_n}\int_{\partial D_r(z)} f(w) d\overline{w}_K \wedge \sum_J \sum_{k \notin J} \sigma(k, J)(\overline{w}_k - \overline{z}_k) d\overline{w}[k, J] \wedge dw \wedge d\overline{z}_J \wedge d\overline{z}_H \quad (2.12)$$

because every term on 2.11 with $I \neq H$ cancels out by repeated wedging, and the factor $(-1)^{p(n-q-1)}\sigma(H)$ assures that dw doesn't appear with the minus sign, regardless of n, p, q and H. Now we consider the summand $J = K$: it equals

$$\frac{1}{nc_n}\sum_{k \notin K}(-1)^{k+1}\frac{1}{r^{2n}}\left[\int_{\partial D_r(z)} f(w)(\overline{w}_k - \overline{z}_k) d\overline{w}[k] \wedge dw\right] \wedge dz_K \wedge d\overline{z}_H$$

because this time $d\overline{w}_K \sigma(k, K) \wedge d\overline{w}[k, K] = (-1)^{k+1} d\overline{w}[k]$ for every k. Now each integral in the sum tends to $c_n \frac{\partial}{\partial \overline{w}_k} f(w)(\overline{w}_k - \overline{z}_k)|_z$ by lemma 2.1.11, so that the sum on all k tends to

$(n-q)f(z)/n \wedge dz_H \wedge d\overline{z}_K$. On the other hand the integrands in 2.12 for each $J \neq K$ have an expression

$$\frac{1}{nc_n}\sum_{k\notin J}\frac{1}{r^{2n}}\int_{\partial D_r(z)} f(w)\sigma(k,J)(\overline{w}_k-\overline{z}_k)d\overline{w}_K\wedge d\overline{w}[k,J]\wedge dw$$

whose terms are nonzero if and only if $K\cup\{k\}=J^c$. But then any such summand is in the form $f(w)(\overline{w}_k-\overline{z}_k)d\overline{w}[k]\wedge dw$, which tends to 0 with r by the integral mean theorem. □

Lemma 2.1.13. *Let U, ω and $D_r(z)$ as in the previous lemma. Then for any $z\in U$*

$$\lim_{r\to 0}\int_{D_r(z)}\omega(w)\wedge\overline{\partial}_z k_{p,q-1}(w,z) = -\frac{q}{n}\,\omega(z)$$

Proof. ω does not depend on z so

$$\omega(w)\wedge\overline{\partial}_z k_{p,q-1}(w,z)=\overline{\partial}_z(\omega(w)\wedge k_{p,q-1}(w,z));$$

moreover we can take the derivatives in and out the integral sign because the resulting forms are still integrable. Take a form as in the previous lemma: we will reason on coefficients, neglecting the differentials in the variables z_i. By definition of $k_{p,q-1}$, for each $j=1,\dots,n$ we must compute the coefficients

$$\frac{\partial}{\partial\overline{z}_j}\int_{D_r(z)} f(w)\frac{(\overline{w}_k-\overline{z}_k)}{\|w-z\|^{2n}}d\overline{w}\wedge dw \tag{2.13}$$

with $k=1,\dots,q$. By using the usual Leibniz rule for differentiation, Stokes Theorem and the fact that $\partial = d$ on $(n-1,n)$ forms it is

$$\int_{D_r(z)} f(w)\frac{(\overline{w}_k-\overline{z}_k)}{\|w-z\|^{2n}}d\overline{w}\wedge dw = \frac{1}{1-n}\int_{D_r(z)} f(w)\frac{\partial}{\partial w_k}\frac{1}{\|w-z\|^{2n-2}}d\overline{w}\wedge dw =$$

$$= \frac{(-1)^{n+k-1}}{1-n}\int_{D_r(z)} d\left(f(w)\frac{d\overline{w}\wedge dw[k]}{\|w-z\|^{2n-2}}\right) - \frac{1}{1-n}\int_{D_r(z)}\frac{\partial}{\partial w_k}f(w)\frac{1}{\|w-z\|^{2n}}d\overline{w}\wedge dw =$$

$$= \frac{(-1)^{n+k-1}}{1-n}\int_{\partial D_r(z)} f(w)\frac{d\overline{w}\wedge dw[k]}{\|w-z\|^{2n-2}} - \frac{1}{1-n}\int_{D_r(z)}\frac{\partial}{\partial w_k}f(w)\frac{1}{\|w-z\|^{2n}}d\overline{w}\wedge dw.$$

Now deriving under integral sign with respect to $\overline{z}_j$, we see that 2.13 is equal to

$$(-1)^{n+k-1}\int_{\partial D_r(z)} f(w)\frac{(w_j-z_j)}{\|w-z\|^{2n}}d\overline{w}\wedge dw[k] + \int_{D_r(z)}\frac{\partial}{\partial w_k}f(w)\frac{(w_j-z_j)}{\|w-z\|^{2n}}d\overline{w}\wedge dw.$$

If $r\to 0$ the first integral converges to $-c_n f(z)\delta_{j,k}$, where $\delta_{j,k}$ is the Kronecker symbol, by lemma 2.1.11, and the second to 0 being integrable in a neighbourhood of z. Now we repeat this a total of q times so, remembering to include the factor $1/nc_n$ from $k_{p,q-1}$, we have the statement. □

Proof of Theorem 2.1.10. Let $\omega \in \mathcal{E}_*^{p,q}(U)$ consists of a single summand; the form $\omega \wedge k_{p,q}$ has no singularities in $U \setminus D_r(z)$ for every $r > 0$ such that $D_r(z) \subset U$. Let $z \in U$; by Stokes formula, the differentiation rule and the fact that ω vanishes on the boundary of U

$$-\int_{\partial D_r(z)} \omega(w) \wedge k_{p,q}(w,z) = \int_{U \setminus D_r(z)} d_w(\omega(w) \wedge k_{p,q}(w,z)) =$$

$$= \int_{U \setminus D_r(z)} \overline{\partial}_w \omega(w) \wedge k_{p,q}(w,z) + \int_{U \setminus D_r(z)} \omega(w) \wedge \overline{\partial}_z k_{p,q-1}(w,z)$$

because by a previous remark $\overline{\partial}_w k_{p,q}(w,z) = (-1)^{p+q} \overline{\partial}_z k_{p,q-1}(w,z)$. Hence by lemmas 2.1.12 and 2.1.13

$$\begin{aligned} \lim_{r \to 0} \quad & \left[\int_{\partial D_r(z)} \omega(w) \wedge k_{p,q}(w,z) + \int_{U \setminus D_r(z)} \omega(w) \wedge \overline{\partial}_z k_{p,q-1}(w,z) \right] = \\ = \quad & \int_U \omega(w) \wedge \overline{\partial}_z k_{p,q-1}(w,z) + \left(1 - \frac{q}{n}\right) \omega(z) + \frac{q}{n} \omega(z) \\ = \quad & \overline{\partial} \int_U \omega(w) \wedge k_{p,q-1}(w,z) + \omega(z) \end{aligned}$$

while for $r \to 0$ the integral $\int_{U \setminus D_r(z)} \overline{\partial}_w \omega(w) \wedge k_{p,q}(w,z)$ converges to the integral over the entire U, being $\overline{\partial}_w \omega(w) \wedge k_{p,q}(w,z)$ integrable around z, whence the thesis. □

This proves that the operator in 2.1 is a homotopy and completes our argument. Accordingly to what has been discussed so far we can restate our conclusions in the hardly-obtained Poincarè Lemma for currents:

Theorem 2.1.14. *Let $T \in \mathcal{D}^{p,q}(U)$, $U \subset \mathbb{C}^n$ regular as in 2.1.10, such that $\overline{\partial} T = 0$. Then for every regular open set $V \subset U$ there exists a current $S \in \mathcal{D}^{p,q-1}(V)$ (defined by 2.10) such that $\overline{\partial} S = T$ on V.*

The technical part is over, and the proof of the main theorem is now at hand.

Proof of Theorem 2.1.2. Consider 2.1. Exactness in $\Omega^{p,0}$ is proposition 2.1.3. For $q > 0$ we must prove that the associated sequence of stalks is exact. To see that take a stalk ω_z in $\mathcal{E}_z^{p,q}$ to be the image of a section $\omega \in \mathcal{E}^{p,q}(U)$ for an open set U containing z and biholomorphic to a disc D in $\mathbb{C}^n$, U small enough to fit into a local chart. Up to composing with a diffeomorphism, we can pretend to be in $\mathbb{C}^n$ and apply proposition 2.1.6. Thus there exist a smaller open set $U' \subset U$ biholomorphic to a smaller disc $D' \subset D$ and $\eta \in \mathcal{E}^{p,q-1}(U')$, such that $\overline{\partial} \eta = \omega$ in U'; the form η projects to a stalk η_z over $\mathcal{E}_z^{p,q-1}$ for which $\overline{\partial} \eta_z = \omega_z$. Exactness on stalks is therefore proved and this is equivalent to exactness of the relative sheaves sequence.

Exactly the same argument applies to the sequence 2.2, this time using corollary 2.1.9 and theorem 2.1.14 instead. To be more explicit regarding the exactness in $\mathcal{D}^{p,0}$, one should first write, on a an open set U biholomorphic to an open set of $\mathbb{C}^n$, a $\overline{\partial}$-closed $(p,0)$ current T as a sum of distributions (viewed as elements of $\mathcal{D}^0$), $T = \sum T_I e_I$. Each T_I is evidently closed, so we can apply corollary 2.1.9 to every one of them. Therefore $T_I = T_{f_I}$ for holomorphic functions f_I, but then $T = T_\omega$ with $\omega = \sum_I f_I dz_I$, and the set of all such $(p,0)$ currents is isomorphic to Ω^p. □

It is worth noting that jointly theorem 2.1.2 and theorem 1.6.4 are nothing but the *Dolbeault Theorem*; restated it reads $H^q(M, \Omega^p) \cong H^{p,q}_{\overline{\partial}}(M)$ for a complex analytic variety M. The de Rham Theorem is similarly obtained if $\mathcal{F}$ is taken to be the constant sheaf resolved by the exact sequence of sheaves of differential k-forms.

What comes next is generalizing these resolutions: call $\mathcal{O}$ the sheaf of holomorphic functions on M and let $\mathcal{F}$ be a *locally free analytic sheaf* on M, that is, the sheaf of holomorphic sections of an analytic vector bundle F on M. It is of course a sheaf of $\mathcal{O}$-modules on M.

Now that we resolved the sheaf of holomorphic p-forms by means of fine sheaves to obtain the Serre duality in its complete generality we want to extend this resolution to the the tensor sheaf $\Omega^p \otimes_{\mathcal{O}} \mathcal{F}$. This is immediately established by observing that the functor $\otimes_{\mathcal{O}}\mathcal{F}$ from the category of sheaves of $\mathcal{O}$-modules on M to itself is exact whenever $\mathcal{F}$ is locally free.

To be more precise let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}$-modules on M of rank k; we denote $\mathcal{E}^{p,q}(\mathcal{F})$ the tensor product of sheaves $\mathcal{E}^{p,q} \otimes_{\mathcal{O}} \mathcal{F}$; the same notation will be used to denote the tensor product with $\mathcal{F}$ of the sheaves $\mathcal{D}^{p,q}$ and Ω^p. Since $\mathcal{F}$ is locally free for the sheaves $\mathcal{E}^{p,q}(\mathcal{F}), \mathcal{D}^{p,q}(\mathcal{F})$ and $\Omega^p(\mathcal{F})$ there exist (*a priori* different) coverings $\{U_\alpha\}$ of M

$$\mathcal{E}^{p,q}(\mathcal{F})|_{U_\alpha} = \bigoplus_{i=1}^{k} \mathcal{E}^{p,q}|_{U_\alpha}, \mathcal{D}^{p,q}(\mathcal{F})|_{U_\alpha} = \bigoplus_{i=1}^{k} \mathcal{D}^{p,q}|_{U_\alpha}, \Omega^p(\mathcal{F})|_{U_\alpha} = \bigoplus_{i=1}^{k} \Omega^p|_{U_\alpha}. \tag{2.14}$$

Sections of these sheaves will be called (p,q) forms (resp. currents, holomorphic p-forms) *with coefficients in* $\mathcal{F}$. Looking back at the exact sequences 2.1 and 2.2 tensorizing by $\mathcal{F}$ and extending properly the differential we have the sequences

$$0 \to \Omega^p(\mathcal{F}) \to \mathcal{E}^{p,0}(\mathcal{F}) \to \mathcal{E}^{p,1}(\mathcal{F}) \ldots \to \mathcal{E}^{p,n}(\mathcal{F}) \to 0. \tag{2.15}$$

and

$$0 \to \Omega^p(\mathcal{F}) \to \mathcal{D}^{p,0}(\mathcal{F}) \to \mathcal{D}^{p,1}(\mathcal{F}) \ldots \to \mathcal{D}^{p,n}(\mathcal{F}) \to 0. \tag{2.16}$$

which are two different fine resolutions of $\Omega^p(\mathcal{F})$. Indeed by using 2.14 one sees that the stalk sequences of the complexes above must be formed by locally free C^∞-modules with exterior differential given by $\bigoplus_k \overline{\partial}_x$ so that exactness comes from exactness on the stalks of 2.1 and

2.2. Moreover these resolutions are fine; we remark that being $\mathcal{E}^{p,q}$ and $\mathcal{D}^{p,q}$ two C^∞-modules they are also $\mathcal{O}$-modules. So, for instance, the morphism of sheaves $g : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q}$ given by multiplication for a differentiable function g extends to a morphism $\overline{g} : \mathcal{E}^{p,q}(\mathcal{F}) \to \mathcal{E}^{p,q}(\mathcal{F})$ by $\overline{g}(\omega \otimes f) = g\omega \otimes f$ which gives to $\mathcal{E}^{p,q}(\mathcal{F})$ the structure of a C^∞-module. If we choose by g a partition of unity the proof is then identical to that of proposition 1.6.2; the same holds for $\mathcal{D}^{p,q}$.

In conclusion we extended theorem 2.1.2 to the sheaf $\Omega^p(\mathcal{F})$, and we shall now focus on the cohomology of these new sequences.

According with the notations of section 1.6.4 for $p \geq 0$ we denote with $H^q(M, \Omega^p(\mathcal{F}))$ the sheaf cohomology of $\Omega^p(\mathcal{F})$ and with $H^q_*(M, \Omega^p(\mathcal{F}))$ the cohomology of $\Omega^p(\mathcal{F})$ with compact support. Instead $H^q(\Gamma_\phi(M, \mathcal{E}^{p,*}(\mathcal{F})))$ and $H^q(\Gamma_\phi(M, \mathcal{D}^{p,*}(\mathcal{F})))$, that is the cohomology groups of the global sections of resolutions 2.15 and 2.16, will be re-written respectively $H^{p,q}(\mathcal{E}_\phi(\mathcal{F}))$ and $H^{p,q}(\mathcal{D}_\phi(\mathcal{F}))$. Again the subscript $*$ stays for cohomology with compact support.

Just as for the Dolbeault Theorem, theorem 1.6.4 applied to resolutions 2.1 and 2.2 yields the following

Theorem 2.1.15. *Let $\mathcal{F}$ be a locally free analytic sheaf on an analytic manifold M, and $\Omega^p(\mathcal{F})$, $\mathcal{E}^{p,q}(\mathcal{F})$, $\mathcal{D}^{p,q}(\mathcal{F})$ as above. There are isomorphisms*

$$H^q(M, \Omega^p(\mathcal{F})) \cong H^{p,q}(\mathcal{E}(\mathcal{F})) \cong H^{p,q}(\mathcal{D}(\mathcal{F}))$$

and

$$H^q_*(M, \Omega^p(\mathcal{F})) \cong H^{p,q}(\mathcal{E}_*(\mathcal{F})) \cong H^{p,q}(\mathcal{D}_*(\mathcal{F})).$$

2.2 Dualities

We are now ready to expose the core results. To do so we must first generalize the topological constructions carried out in 1.4 for forms of degree k with coefficients in $\mathbb{C}$, to (p, q) differential forms with coefficients in an arbitrary locally free analytic sheaf $\mathcal{F}$. Let M be a complex analytic manifold. Write, for multi-indices $S = (s_1, \ldots, s_p)$, $T = (t_1, \ldots, t_q)$ of arbitrary length p and q, possibly with repeated elements, $\partial^{S,T} = \partial^{p+q}/\partial z_{s_1} \ldots \partial z_{s_p} \partial \overline{z}_{t_1} \ldots \partial \overline{z}_{t_q}$. This time we consider over the complex vector space $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$ the seminorms

$$\rho_{K,S,T}(\omega) = \sup_{r,IJ} \sup_{z \in K} |\partial^{S,T} f_{r,IJ}(z)| \tag{2.17}$$

for K compact in M. This time we are not asking ω to be supported in K or compactly supported at all. The coefficients $f_{r,IJ}$ are to be understood as follows: choose a local chart covering $\{U_\alpha\}$ of M as in 2.14. Over each U_α, by the locally free structure of $\mathcal{E}^{p,q}(\mathcal{F})$, ω expresses as a sum of $\omega(z) = \omega_{\alpha_1}(z) + \ldots + \omega_{\alpha_k}(z)$ for certain (p, q) differential forms ω_{α_r},

$r = 1, \dots k$, on U_α, each one having $\binom{n}{p}\binom{n}{q}$ local coefficients $f_{r,IJ}$. As in section 1.4 for each z in K the supremum must also include the fact that there may be different local chart expression $f_{r,IJ}$ of ω_r on the U_αs overlapping around z; as in that case these are in a finite number for all $z \in K$ because by compactness of K we can pick up a finite number of U_α covering K. This means that the supremum above is well defined.

The family $\{\rho_{K,S,T}\}$ is a separating family of seminorms generating a locally convex Hausdorff topology on $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$: as prefigured in section 1.4 this topology is metrizable. Indeed let $\{K_n\}$ be a sequence of compact sets such that $K_n \subset K_{n+1}$ and $M = \bigcup_n K_n$. Let $X = \Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$; the ball $B_{r,K,S,T}(0) = \{\omega \in X | \, \rho_{K,S,T}(\omega) < r\}$ of radius r is an open neighbourhood of 0, let n such that $K \subset K_n$. Then the neighbourhood $B_{1/n,K_n,S,T}(0) = \{\omega \in X | \, \rho_{K_n,S,T}(\omega) < 1/n\}$ of 0 clearly contains $B_{1/n,K,S,T}(0)$. Hence $\{\rho_{K_n,S,T}\}$ induces the same fundamental neighbourhood system of 0 of $\{\rho_{K,S,T}\}$ so that $\rho_{n,S,T} := \rho_{K_n,S,T}$ is a countable family of seminorms generating the same topology of $\rho_{K,S,T}$ and metrizability follows by proposition 1.3.3.

This topology, denoted ς, is the *topology of the local uniform convergence* on $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$: indeed a sequence $\omega^i \to 0$ tends to 0 in ς if and only if every local coefficient $\partial^{S,T} f^i_{r,IJ}$ tends to 0 uniformly on every compact set.

Proposition 2.2.1. $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$ *with the topology* ς *is a Fréchet space.*

Proof. We just have to prove that it is complete. This is quite immediate because convergence in our topology is basically uniform convergence. Take a Cauchy sequence ω^n and write it locally on a compact K as in 2.17; the derivative $\partial^{S,T} f^n_{r,IJ}$ of every coefficient converges uniformly on every $K \subset U_\alpha$ to a smooth function $g^{S,T}_{r,IJ}$. In particular by the theorem of regularity of the uniform limit of a sequence of derivatives we have that the limit $g_{r,IJ}$ of the functions $f^n_{r,IJ}$ is smooth and such that $\partial^{S,T} g_{r,IJ} = g^{S,T}_{r,IJ}$. Hence $\omega^n \to \omega$ where ω is the form with coefficients in $\mathcal{F}$ locally given by the functions $g_{r,IJ}$.

□

Now we want to describe the dual space of the topological vector space $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$ with respect to the local uniform convergence topology. We will prove that this time the linear continuous functionals on this space can be represented by compactly supported *currents* with coefficients in the dual sheaf $\mathcal{F}^*$ of $\mathcal{F}$. More precisely the theorem is the following

Theorem 2.2.2. *Let* M *be an analytic manifold of dimension* n *and* $\mathcal{V}$ *a locally free analytic sheaf on* M*. The topological dual of the Fréchet space* $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{V}))$*, with the weak-* topology, is isomorphic to* $\Gamma(M, \mathcal{D}^{n-p,n-q}_*(\mathcal{F}^*))$.

One can find this result for distributions and functions of real variable on the groundbreaking book of Schwartz [21]; the theorem in its full generality is an extension of Serre in

[24]. The statement can be further generalized to any sheaf $\mathcal{V}$ of C^∞-modules but this case will not be treated. The key to the proof is reducing ourselves, via a partition of unity, to open sets where the forms involved are isomorphic a sum of forms with coefficients in $\mathbb{C}$. There the dual space is just the space of currents; the argument is indeed very similar to that of the Poincaré Duality Theorem.

Proof. Once an isomorphism is given bicontinuity follows by endowing both spaces with the weak-* topology; we therefore only need to exhibit an algebraic linear 1-1 map. A little preamble to the proof illustrating how the duality is constructed is necessary.

Let $\mathcal{V}$ be a finite rank vector bundle on M and consider its dual bundle $\mathcal{V}^*$; the canonical bilinear form on the product $\mathcal{V}_z \times \mathcal{V}_z^*$ yields to a morphism of the associated sheaves of sections $\mathcal{V} \otimes_{\mathcal{O}} \mathcal{V}^* \to \mathcal{O}$, defined stalkwise by $v \otimes V \mapsto f$ with $f(z) = V(z)(v(z))$ for $z \in U \subset M$, $v \in \Gamma(U, \mathcal{V})$, $V \in \Gamma(U, \mathcal{V}^*)$.

On the other hand we can define an exterior product between forms with arbitrary support and currents with compact support exactly like in section 1.4, which this time produces compactly supported currents. In other words there is a morphism of sheaves

$$\begin{aligned} \wedge : \Gamma(M, \mathcal{E}^{p,q}) \otimes_{\mathcal{O}} \Gamma(M, \mathcal{D}_*^{n-p,n-q}) &\longrightarrow \Gamma(M, \mathcal{D}_*^{n,n}) \\ \eta \otimes S &\longmapsto \eta \wedge S \end{aligned} \tag{2.18}$$

By tensoring $\mathcal{E}^{p,q}$ with $\mathcal{V}$ and $\mathcal{D}_*^{n-p,n-q}$ with $\mathcal{V}^*$ we can extend 2.18 to a new morphism, again called $\wedge$, by observing that for η and S with coefficients in $\mathbb{C}$, $v \in \Gamma(M, \mathcal{V})$ and $V \in \Gamma(M, \mathcal{V}^*)$ the element

$$(\eta \otimes v) \wedge (S \otimes V) = (\eta \wedge S) \otimes (v \otimes V) \tag{2.19}$$

is still in $\Gamma(M, \mathcal{D}_*^{n,n})$ since on the right side we are tensoring with an element in $\mathcal{O}$. We call $\omega = \eta \otimes v$ and $T = S \otimes V$ and we define the expression 2.19 to be the image of the morphism

$$\begin{aligned} \wedge : \Gamma(M, \mathcal{E}^{p,q}(\mathcal{V})) \otimes_{\mathcal{O}} \Gamma(M, \mathcal{D}_*^{n-p,n-q}(\mathcal{V}^*)) &\longrightarrow \Gamma(M, \mathcal{D}_*^{n,n}) \\ \omega \otimes T &\longmapsto \omega \wedge T. \end{aligned} \tag{2.20}$$

We can now integrate the compactly supported distribution $\omega \wedge T$; from 2.20 we deduce the bilinear form

$$\begin{aligned} \int : \Gamma(M, \mathcal{E}^{p,q}(\mathcal{V})) \times \Gamma(M, \mathcal{D}_*^{n-p,n-q}(\mathcal{V}^*)) &\longrightarrow \mathbb{C} \\ (\omega, T) &\longmapsto \int_M \omega \wedge T. \end{aligned} \tag{2.21}$$

Fixing T we obtain a linear continuous functional $\int^T : \Gamma(M, \mathcal{E}^{p,q}(\mathcal{V})) \to \mathbb{C}$, that is , an element of $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{V}))^*$: our claim is that the linear mapping $T \to \int^T$ is the required isomorphism of dual spaces.

First of all of course $T = 0$ if and only if $\int^T \equiv 0$; we have then only to show that $\int^T$ is continuous, and that every continuous funtional L on $(\mathcal{E}^{p,q}(\mathcal{V}), \varsigma)$ is of the form $\int^T$ for a certain compactly supported $(n-p, n-q)$ current T with coefficients in $\mathcal{V}$. We begin from the second.

Choose on M a locally finite covering $\{U_\alpha\}$ over which the sheaf $\mathcal{V}$ trivializes and such that every U_α is relatively compact; let k be the rank of $\mathcal{V}$.

If ω^n is an arbitrary sequence converging to 0 in ς and having support in a compact $K \subset U_\alpha$. By the locally free structure of $\mathcal{V}$ up to an isomorphism $\omega^n = \sum_{i=1}^k \omega_i^n$, with $\omega_i^n \in \Gamma(U_\alpha, \mathcal{E}_*^{p,q})$, all ω_i^n having support in K. Then by proposition 1.4.1 for all i we have that ω_i^n converges to 0 in the topology τ of $\Gamma(U_\alpha, \mathcal{E}_*^{p,q})$ defined in 1.4. Call ϕ_α the isomorphism

$$\bigoplus_{i=1}^k \Gamma(U_\alpha, \mathcal{E}_*^{p,q}) \cong \Gamma(U_\alpha, \mathcal{E}_*^{p,q}(\mathcal{V})|_{U_\alpha})$$

and j_i the immersion of the i-th summand. Now the continuous $L_{\alpha_i} = L|_{U_\alpha} \circ \phi_\alpha \circ j_i$ in ς is, by the previous reasoning, a sequentially continuous function from $(\Gamma(U_\alpha, \mathcal{E}_*^{p,q}), \tau)$ to $\mathbb{C}$, therefore continuous by [20] theorem 6.6 (or the remark in section 1.4).

Then by definition there exist unique currents $T_{\alpha_i} \in \Gamma(U_\alpha, \mathcal{D}^{n-p,n-q})$ such that $L_{\alpha_i} = T_{\alpha_i}$; then $T_\alpha = \bigoplus_{i=1}^k T_{\alpha_i}$ is, up to an isomorphism, an element of $\Gamma(U_\alpha, \mathcal{D}^{p,q}(\mathcal{V}^*)|_{U_\alpha})$ such that $T_\alpha = L|_{U_\alpha}$. By uniqueness of the T_{α_i}s the collection of sections $\{T_\alpha\}$ satisfies $T_\alpha = T_\beta$ on $U_\alpha \cap U_\beta$, so by the sheaf axiom it glues together to a global section T. Let now $\omega \in \mathcal{E}^{p,q}(\mathcal{V})$ and select a C^∞ partition of unity $\{\rho_\alpha\}$ subordinated to $\{U_\alpha\}$ it is

$$L(\omega) = \sum_\alpha L(\rho_\alpha \omega) = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega \wedge T_\alpha = \int_M \omega \wedge T = \int^T (\omega). \tag{2.22}$$

Finally T has compact support. To prove that, it is sufficient to check that $T_\alpha = 0$ for every α except a finite number: if this were the case then let α_i, $i = 1, \ldots, n$, such that T_{α_i} is not identically 0; for every ω with compact support if $K = \bigcup_i \overline{U}_{\alpha_i}$ and $\mathrm{Supp}\omega \subset M \setminus K$ we have $T(\omega) = 0$ so that T has compact support. In our case this happens because if the sums in 2.22 were infinite then one could choose ω to have constant coefficients so that the series diverges, negating the first equality.

Let us show that conversely any functional $\int^T$ is continuous. Let ω^n be a sequence converging to 0 in the topology ς of $\mathcal{E}^{p,q}(\mathcal{V})$. Then for every α it is $\mathrm{Supp}\rho_\alpha\omega^n \subset K \subset U_\alpha$ for all n; again there $\rho_\alpha \omega^n = \sum_i \omega_{\alpha_i}^n$ and

$$\int_M \omega^n \wedge T = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega^n \wedge T = T\left(\sum_{\alpha,i} \omega_{\alpha_i}^n\right);$$

if ω^n tends to 0 in ς then $\sum_{\alpha,i} \omega_{\alpha_i}^n \to 0$ in τ for all α; but T is continuous in τ so the left side tends to 0.

$\square$

From now on we identify the topological dual of the (p,q) forms with coefficients in a locally free analytic sheaf $\mathcal{V}$ with $\Gamma(M, \mathcal{D}_*^{n-p,n-q}(\mathcal{V}^*))$ without further mention. Next we focus on the adjoint operator of the differential $\overline{\partial}$.

Proposition 2.2.3. *The adjoint operator of the continuous linear map*

$$\overline{\partial} : \Gamma(M, \mathcal{E}^{p,q}(\mathcal{V})) \longrightarrow \Gamma(M, \mathcal{E}^{p,q+1}(\mathcal{V}))$$

is the continuous linear map

$$(-1)^{p+q+1}\overline{\partial} : \Gamma(M, \mathcal{D}_*^{n-p,n-q-1}(\mathcal{V}^*)) \longrightarrow \Gamma(M, \mathcal{D}_*^{n-p,n-q}(\mathcal{V}^*)).$$

Proof. The continuity of $\overline{\partial}$ on forms is obvious by definition; in fact if every derivative of every coefficient of a sequence ω^n converges uniformly to 0 so will every derivative of every coefficient of $\overline{\partial}\omega^n$, simply because its coefficients are derivatives of those of ω^n. If $T_n \to 0$ and every T_n is compactly supported and defined over $\Gamma(M, \mathcal{E}^{p,q+1}(\mathcal{V}))$ then in particular for all forms $\omega \in \Gamma(M, \mathcal{E}^{p,q}(\mathcal{V}))$, it is $T_n(\overline{\partial}\omega) \to 0$, thus $\overline{\partial}T_n \to 0$, so $\overline{\partial}$ is continuous with respect to the weak-* topology on currents. Now let ω be a (p,q) form with coefficients in $\mathcal{V}$ and $T \in \Gamma(M, \mathcal{D}_*^{n-p,n-q-1}(\mathcal{V}^*))$: we have, by identifiying ω with its representation as a current, and observing that $\overline{\partial} = d$ on $(n, n-1)$ currents

$$d(\omega \wedge T) = \overline{\partial}(\omega \wedge T) = \overline{\partial}\omega \wedge T + (-1)^{p+q}\omega \wedge \overline{\partial}T.$$

On the other hand it is $\int_M d(\omega \wedge T) = \omega \wedge T(d1) = 0$ whence

$$\int_M \omega \wedge \overline{\partial}T = (-1)^{p+q+1} \int_M \overline{\partial}\omega \wedge T$$

and the proposition follows. □

Now that we established a duality between two cochain complexes we must switch to cohomology groups: in the category of topological vector spaces these are quotient vector spaces endowed with the quotient topology. Unfortunately cohomology in this category is often ill-behaved: we must then make additional hypotheses on our complexes if we want to go further.

We start out introducing a new definition, that of *strict morphism* of topological vector spaces; what we will see is that for the Duality Theorem to hold the differentials $\overline{\partial}$ are required to have this additional property. In general a linear continuous map $f : X \to Y$ between topological vector spaces can be factored in

$$X \xrightarrow{\pi} X/\mathrm{ker}f \xrightarrow{\overline{f}} f(X) \xrightarrow{i} Y \tag{2.23}$$

where π and i are respectively the projection on the quotient and the canonical injection. In this decomposition the continuous isomorphism $\overline{f}([x]) = f(x)$ in general is not a homeomorphism;

for instance if X is not a discrete set and X is endowed with the discrete topology $\overline{f} = f$ cannot be bicontinuous, provided Y does not have the same topology [1]

Definition 2.2.4. Let X and Y be topological vector spaces and $f : X \to Y$ a morphism between them, that is, a continuous homomorphism; f is said to be a *strict morphism* or a *topological homomorphism* if $\overline{f}$ in 2.23 is a homeomorphism.

This is essential to the Duality: it is required in the next proposition which is the fact from topological vector space theory providing the final argument for the Theorem.

Proposition 2.2.5. *Let L, M, N be three Fréchet spaces, u and v two strict morphisms*

$$L \xrightarrow{u} M \xrightarrow{v} N$$

such that $v \circ u = 0$. Call L^*, M^*, N^* the respective dual spaces with respect to a separating dual operator $<,>$, with ${}^tu$, ${}^tv$ the transposed morphisms of u and v. Then $H = \dfrac{ker v}{u(L)}$ is a Fréchet space and the dual complex

$$N^* \xrightarrow{{}^tv} M^* \xrightarrow{{}^tu} L^*$$

is such that $H' = \dfrac{ker^t u}{{}^tv(N^*)}$ is isomorphic to the topological dual H^* of H.

Remark. The mild assumption that $<,>$ is *separating* (i.e. not degenerate in both variables) assures that the the dual spaces above are not trivial, circumstance that may otherwise happen (see [3] chap. 2). Anyway this will not be the case when $<,>$ is the bilinear form defined in theorem 2.2.2.

Proof. Some notations: we set $C = \ker v$, $B = u(L)$, $C' = \ker^t u$, $B' = {}^tv(N^*)$ so that $H = C/B$ and $H' = C'/B'$. First of all

$$N^* \xrightarrow{{}^tv} M^* \xrightarrow{{}^tu} L^*$$

is such that ${}^tu \circ {}^tv = 0$. Indeed $\forall d \in N^*$, $x \in L$

$$< {}^tu \circ {}^tv(d), x >=< d, v \circ u(x) >= 0$$

Since v is continuous $v^{-1}(0)$ $=\ker v$ is closed, therefore complete so C is a Fréchet space. Moreover being u a strict morphism we have that $B = u(L) \cong L/\ker u$ is also Fréchet by 1.3.4, being $\ker u$ closed. In particular B is complete, hence closed in C; again by proposition 1.3.4 we have that H is a Fréchet space.

[1] This is due to the fact that the category of topological vector spaces is not abelian. Indeed the arrow f between X and Y is a monomorphism and an epimorphism but it is *not* an isomorphism. For clarifications see [19] chap.7

For each coset $h' \in H'$ take as $c' \in C'$ an element representing h'; the set

$$\begin{aligned} C' &= \{m \in M^* | < {}^t u(m), x >= 0, \forall x \in L\} \\ &= \{m \in M^* | < m, u(x) >= 0, \forall x \in L\} \\ &= \{m \in M^* | m \ \equiv 0 \text{ on } B\} \end{aligned} \tag{2.24}$$

is in 1-1 correspondence with the linear continuous functionals on the quotient M/B. This means that by restricting c' to C we have a well defined continuous linear functional

$$\begin{aligned} \overline{c}' : H &\longrightarrow \mathbb{C} \\ [c] &\mapsto < c', c > . \end{aligned} \tag{2.25}$$

The rule $h' \mapsto \overline{c}'$ will be our morphism of topological vector spaces; it is clearly linear. We must prove that it depends only on h', that is injective, surjective and continuous.

The first two are equivalent to $\overline{c}' = 0$ if and only if $h' = 0$. If $\overline{c}' = 0$ then $c' \equiv 0$ on C; being v a strict morphism c' descends uniquely to a continuous linear functional c'_0 from $v(M) \cong M/C$ which acts like $< c'_0, y >=< c', x >$ for x in M and $y = v(x)$. Now choose a continuous linear functional d that continuously extends c'_0 on N; we have for every $x \in M$

$$< {}^t v(d), x >=< d, v(x) >=< c'_0, v(x) >=< c', x >$$

so $c' \in B'$ and thus $h' = 0$.

All of the implications above reverse so that the isomorphism is well defined; more precisely if h' is the 0 coset it can be represented by an element c' in B' so there is a functional $d \in M^*$ which is its transpose by ${}^t v$, that is, d restricted to $v(M)$ acts like c'; but v is strict so d, and thus c', defines a continuous linear functional from M/C, which implies $c' = 0$ on C, whence $\overline{c}' = 0$.

To prove the surjectivity take a continuous linear functional λ on H: there exists $c' \in C^*$ vanishing on B such that $\lambda([c]) =< c', c >$ for all c in C. By the Hahn-Banach theorem c' extends to a continuous linear functional c'' on the whole M that still vanishes on B, which by 2.24 projects on the quotient to $\overline{h}'$ on H' so that finally $\overline{h}'([c]) =< c'', c >=< c', c >= \lambda([c])$. □

It is time to gather everything together and state the fundamental theorem of this thesis. It is now just an easy consequence of the theorems proved throughout in sections 2.1 and 2.2.

Theorem 2.2.6 (Serre Duality Theorem, 1955). *Let M be a complex analytic manifold and $\mathcal{F}$ a locally free analytic sheaf on M. If the sequence of sheaves*

$$\Gamma(M, \mathcal{E}^{p,q-1}(\mathcal{F})) \xrightarrow{\overline{\partial}} \Gamma(M, \mathcal{E}^{p,q}(\mathcal{F})) \xrightarrow{\overline{\partial}} \Gamma(M, \mathcal{E}^{p,q+1}(\mathcal{F})) \tag{2.26}$$

is such that $\overline{\partial}$ is a strict morphism then the topological dual of the Fréchet space $H^q(M, \Omega^p(\mathcal{F}))$ is isomorphic to $H^{n-q}_*(M, \Omega^{n-p}(\mathcal{F}^*))$.

Proof. Take in proposition 2.2.5

$$L = \Gamma(M, \mathcal{E}^{p,q-1}(\mathcal{F})),\ M = \Gamma(M, \mathcal{E}^{p,q}(\mathcal{F})),\ N = \Gamma(M, \mathcal{E}^{p,q+1}(\mathcal{F})),$$

$u, v = \overline{\partial}$ and $<,>$ given by the integral in 2.21. By theorem 2.2.2 we have

$$L^* = \Gamma(M, \mathcal{D}_*^{n-p,n-q+1}(\mathcal{F}^*)),\ M^* = \Gamma(M, \mathcal{D}_*^{n-p,n-q}(\mathcal{F}^*)),\ N^* = \Gamma(M, \mathcal{D}_*^{n-p,n-q-1}(\mathcal{F}^*))$$

and by proposition 2.2.3 ${}^t v = (-1)^{p+q+1}\overline{\partial}$ and ${}^t u = (-1)^{p+q}\overline{\partial}$ so that $H = H^{p,q}(\mathcal{E}(\mathcal{F}))$ and $H' = H^{n-p,n-q}(\mathcal{D}_*(\mathcal{F}^*))$. Therefore theorem 2.1.15 shows that $H \cong H^q(M, \Omega^p(\mathcal{F}))$ and $H' \cong H_*^{n-q}(M, \Omega^{n-p}(\mathcal{F}^*))$ thus finally

$$H^q(M, \Omega^p(\mathcal{F}))^* \cong H_*^{n-q}(M, \Omega^{n-p}(\mathcal{F}^*))$$

by using proposition 2.2.5. □

This duality can also be regarded topologically by endowing $H^q(M, \Omega^p(\mathcal{F}))^*$ with the weak-* topology.

The case $p = 0$ is the most used in applications and stands out as a corollary, simply reminding definition 1.1.4:

Corollary 2.2.7. *Let M, $\mathcal{F}$ and $\overline{\partial}$ be as in theorem 2.2.6 and let $p = 0$ in 2.26. We have*

$$H^q(M, \mathcal{F})^* \cong H_*^{n-q}(M, \mathcal{K}(\mathcal{F}^*))$$

where $\mathcal{K}$ is the canonical sheaf over M.

2.3 Examples, Counterexamples and Applications

Here we illustrate some important facts relating to the Serre Duality as well as some results following from it.

2.3.1 A Sufficient Condition for $\overline{\partial}$ to Be a Strict Morphism

Proposition 2.3.1. *Let M, $\mathcal{F}$ as in Theorem 2.2.6. If $dim H_{\mathbb{C}}^q(X, \Omega^p(\mathcal{F})) < \infty$ then for every q*

$$\overline{\partial} : \Gamma(M, \mathcal{E}^{p,q-1}(\mathcal{F})) \longrightarrow \Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$$

is a strict morphism.

Proof. Set $L = \Gamma(M, \overline{\partial}(\mathcal{E}^{p,q-1}(\mathcal{F}))$; by theorem 2.1.15 the hypothesis is equivalent to say that the codimension of $\overline{\partial}(L)$ in ker$\overline{\partial}$ is finite. Hence exists a finite dimensional subspace M of ker$\overline{\partial}$ such that, at least algebraically, ker$\overline{\partial} \cong \overline{\partial}(L) \oplus M$: define

$$\begin{array}{rcl} f : L \times M & \longrightarrow & \ker\overline{\partial} \subset \Gamma(M, \mathcal{E}^{p,q}(\mathcal{F})) \\ (x, y) & \longmapsto & \overline{\partial}(x) + y \end{array}$$

f is linear, continuous and surjective. M is finite dimensional thus metrizable and complete so that $L \times M$ is metrizable and complete; the same holds for ker$\overline{\partial}$ as is a Fréchet space, being a closed subspace of the Fréchet space $\Gamma(M, \mathcal{E}^{p,q}(\mathcal{F}))$ by the continuity of $\overline{\partial}$. Therefore by Banach's Theorem ([3] chap. 2 p. 17) f is a strict morphism whence, as topological vector spaces, $(L/\ker\overline{\partial}) \times M \cong \overline{\partial}(L) \oplus M$ by the map $([x], y) \mapsto \overline{\partial}(x) + y$ which implies $L/\ker\overline{\partial} \cong \overline{\partial}(L)$ and $\overline{\partial}$ is itslef strict. □

2.3.2 The Theorem for Compact Manifolds

If M is compact then any locally free holomorphic sheaf on M is such that $H^q(X, \mathcal{F})$ has finite dimension (for a proof see [25]); moreover every section of any bundle over M has compact support, thus $H^q_*(X, \mathcal{F}) \cong H^q(X, \mathcal{F})$. Using the previous result and theorem 2.2.6

$$H^q(M, \Omega^p(\mathcal{F})) \cong H^q(M, \Omega^p(\mathcal{F}))^* \cong H^{n-q}(M, \Omega^{n-p}(\mathcal{F}^*))$$

for any locally free sheaf $\mathcal{F}$ on M. In particular these vector spaces have same dimension and for $p = 0$

$$H^q(M, \mathcal{F})^* \cong H^q(M, \mathcal{F}) \cong H^{n-q}(M, \mathcal{K}(\mathcal{F}^*))$$

2.3.3 Stein Manifolds and a Counterexample

The Duality takes a remarkable form also when applied to a certain class of complex manifolds whose properties make them look like analytic versions of affine algebraic varieties. These manifolds are called Stein manifolds; a Stein manifold is a manifold that possesses many globally defined holomorphic functions. An extensive treatment of these objects, that actually requires a deep knowledge of analytic properties of functions of several complex variables, can be found in [13].

Definition 2.3.2. A *Stein* manifold is an analytic complex manifold M of dimension n such that

1. M is *holomorphically convex*, that is for each compact $K \subset M$ the set $\overline{K} = \{z \in M, |f(z)| \leq \sup_K |f|, \forall f \in \mathcal{O}_X\}$ is compact;

2. For every $z, w \in M$ there exists a holomorphic function f on M such that $f(z) \neq f(w)$;

3. For every point $z \in M$ there are n holomorphic functions on M generating the tangent space at z.

It follows from the definition that M has nonconstant holomorphic functions, so it cannot be compact. A closed submanifold of a Stein manifold is a Stein manifold, and a Stein manifold of dimension n can always be biholomorphically embedded in $\mathbb{C}^{2n+1}$: this means that any Stein manifold can be realized as a (particular) closed submanifold of a complex affine space. Examples of Stein manifolds are $\mathbb{C}^n$, open discs in $\mathbb{C}^n$, noncompact Riemann Surfaces, domains of holomorphy (see definition 2.3.5). Since this definition is given only to motivate a counterexample we do not go into the many important geometric properties of this class of manifolds. Basically what we need is the following important cohomological result on *coherent sheaves* on Stein manifolds, due to Cartan.

Theorem 2.3.3 (Cartan's B Theorem, 1951). *Let M be a Stein manifold and $\mathcal{F}$ a coherent analyic sheaf on M. Then $H^q(X, \mathcal{F}) = 0$ for $q > 0$.*

Proof. See [6] exposé 19. □

In view of this theorem we can state the Duality for Stein manifolds.

Theorem 2.3.4. *Let M be a Stein Manifold and $\mathcal{F}$ an analytic locally free sheaf on M. Then for every p*

$$H^q_*(M, \Omega^p(\mathcal{F})) = 0$$

for $q \neq n$, and

$$H^n_*(M, \Omega^p(\mathcal{F})) \cong H^0(M, \Omega^{n-p}(\mathcal{F}^*))^*$$

Proof. By theorem 2.3.3 all cohomology groups $H^q(M, \Omega^p(\mathcal{F})$ are zero for $q > 0$, so all intermediate cohomology groups vanish and by 2.3.1 every $\overline{\partial}$ is a strict morphism. Being $\mathcal{F}$ locally free it is also coherent so we just apply theorem 2.2.6 with $\mathcal{F}^*$ in place of $\mathcal{F}$, $q = 0$ and $n - p$ in place of p. □

We must now turn our attention to a special class of domains of $\mathbb{C}^n$ whose nature is related to the extendability of holomorphic functions to a larger domain. Such domains are called *domains of holomorphy*. A domain of holomorphy is basically a domain of $\mathbb{C}^n$ maximal with respect to extension of any holomorphic function to a larger set, that is to say a domain D where no kind of extension theorems holds for at least a holomorphic function. More precisely

Definition 2.3.5. An open connected set $D \subset \mathbb{C}^n$ is called a *domain of holomorphy* if for any $z \in \partial D$ there is an open neighbourhood U of z and a holomorphic function f on $V = U \cap D$ such that does not exists a holomorphic function $\overline{f}$ extending f on U.

For $n = 1$ this definition is uninteresting since every domain in $\mathbb{C}$ is of holomorphy; for every z on the boundary it is sufficient to take f holomorphic whose zeros accumulate on z: by the identity theorem of analytic functions of one variable this function cannot be extended on a neighbourhood of z.

To see how domains which are not of holomorphy look like we may start with considering a subset $X \subset D \subset \mathbb{C}^n$ of a domain D, such that for every $z \in X$ there exist a disc $D_r(z)$ and a holomorphic function on D such that $f = 0$ on $X \cap D_r(z)$. Such a set X is called a *thin subset* of D. The following theorem holds

Theorem 2.3.6 (Riemann Extension Theorem). *Let $D \subset \mathbb{C}^n$ be a domain, $X \subset D$ thin, and f a locally bounded holomorphic function on $D \setminus X$. Then f extends to a holomorphic function $\overline{f}$ on all D.*

Proof. See [13] p. 19. □

Hence $D \setminus X$ is not a domain of holomorphy. Other examples of domains that are not of holomophy are circular shells, punctured n-complex spaces with $n \geq 2$ or the complementary of a compact set (by Hartog's Theorem, see [13] p. 228). The counterexample we are looking for is to be found in domains that are *not* domains of holomorphy.

Consider in $\mathbb{C}^2$ the real line $R = \{\mathrm{Im} z_1 = \mathrm{Im} z_2 = \mathrm{Re} z_2 = 0\}$. The function $z_1 z_2$ is holomorphic in every neighbourhood of every point of R and vanishes on R; therefore R is thin and by the Extension Theorem $X = \mathbb{C}^n \setminus R$ is not a domain of holomorphy.

Let us now examine the exact sequence of sheaves of holomorphic functions

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\mathbb{C}^2} \longrightarrow \mathcal{O}_R \to 0$$

where $\mathcal{O}_X$ denotes the 0 extension outside X of the sheaf of the holomorphic functions of X. By applying the long exact sequence of cohomology with compact support we have

$$\ldots \longrightarrow H^0_*(R, \mathcal{O}_R) \longrightarrow H^1_*(X, \mathcal{O}_X) \longrightarrow H^1_*(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}) \to \ldots$$

R is not compact, so by the analytic continuation theorem there are no analytic functions with compact support on R, thus $H^0_*(R, \mathcal{O}_R) = 0$; moreover being $\mathbb{C}^2$ a Stein manifold $H^1_*(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2}) = 0$ by theorem 2.3.4: in conclusion it is also $H^1_*(X, \mathcal{O}_X) = 0$. From this we deduce $H^1_*(X, \Omega^2_X) = 0$ because on X every holomorphic 2-form admits a global expression and by taking its coefficient we establish an isomorphism $\mathcal{O}_X \cong \Omega^2_X$.

Now the fact that X is not a domain of holomorphy comes into play. Namely suppose to have an open set D of $\mathbb{C}^n$ such that $H^q(D, \mathcal{O}_D) = 0$ for $q = 1, \ldots, n-1$: then by an argument due to Cartan cited by Serre in [22], note 7, D is necessarily a domain of holomorphy. Since X is not a domain of holomorphy we conclude $H^1(X, \mathcal{O}_X) \neq 0$ so that $H^1(X, \mathcal{O}_X) \neq H^1_*(X, \Omega^2_X)$ and Serre Duality Theorem does not hold for X. Since all other hypotheses are verified, this may only be due to the fact that some of the maps $\overline{\partial}$ fail to be a strict morphism.

2.3.4 Divisors on Compact Riemann Surfaces

We study now the case of complex compact connected analytic manifolds M of dimension 1, that is compact connected Riemann Surfaces, and explain the duality in this context.

A divisor on a Riemann Surface is a formal finite sum $D = \sum_i n_i p_i$ of points $p_i \in M$ with integer coefficients, that is to say an element of the free $\mathbb{Z}$-module generated by points of M. The union of the p_is with $n_i \neq 0$ is the *support* of the divisor. A *meromorphic function* is an assignment of an open covering $\{U_\alpha\}$ on M with a collection of fractions $f_\alpha = g_\alpha/h_\alpha$, g_α, h_α holomorphic on U_α such that $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$. A divisor is *principal* if it is the set of zeros or poles of a meromorphic function counted with their multiplicity, effective if $n_p \geq 0$ for all p.

We used throughout the fact that there is an equivalence between the categories of vector bundles of rank k on a manifold M and locally free sheaves of $\mathcal{O}_M$-modules of rank k; when $k = 1$ both classes, that of *line bundles* over M and of *invertible sheaves* of $\mathcal{O}_M$-modules, have a group structure which is preserved under this equivalence. Divisors on a Riemann Surface have a representation in terms of both of these classes; conversely the fact that a line bundle (thus a locally free sheaf) represents a divisor is related to the existence of meromorphic sections of such a bundle. In what follows we explain this connection.

Let $\mathcal{M}$ be the sheaf of holomorphic functions on S, $\mathcal{M}^* = \mathcal{M} \setminus \{0\}$ and $\mathcal{O}^*$ the sheaf of never vanishing holomorphic functions. The group $\mathrm{Div}(S)$ of divisors on a compact connected Riemann surface S has the following cohomological description

Proposition 2.3.7. $\mathrm{Div}(S) \cong H^0(S, \mathcal{M}^*/\mathcal{O}^*)$.

Proof. There is an exact sequence of sheaves

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$$

Choose a finite covering $\{U_\alpha\}$ of S; an element s in $H^0(M, \mathcal{M}^*/\mathcal{O}^*)$ is represented by a 0-cocycle in $f = \{f_\alpha \in \mathcal{M}^*(U_\alpha)\}$ subject to the condition $f_\alpha/f_\beta = g_{\alpha\beta}$ in $U_\alpha \cap U_\beta$ for a certain function $g_{\alpha\beta} \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$, for all α, β. This means that if $p \in U_\alpha \cap U_\beta$ and n_p is the order of pole or 0 of f_α it is also the order of f_β at p. So we let the divisor of f to be the well defined divisor $\mathrm{div}(f) = \sum_p ord_p f_\alpha p$, whichever U_α contains p, the sum being finite because the covering is finite. Conversely given $D = \sum_i n_i p_i$ let $\{U_\alpha\}$ be taken in such a way that each U_α contains at most one p_i, and let g_α be the meromorphic function $g_\alpha = \prod_i (z_\alpha - p_i)^{n_{p_i}}$ for a local coordinate z_α on U_α. When z_α ranges over $\{U_\alpha\}$ we see that the collection $g = \{g_\alpha\}$ satisfies $g_\alpha/g_\beta \in \mathcal{O}^*_{U_\alpha \cap U_\beta}$ and $\mathrm{div}(g) = D$. This correspondance is actually 1-1 because if two distinct sets $\{f_\alpha\}$, $\{g_\alpha\}$ have the same divisor, differing on intersections by multiplcation by a holomorphic function, they are in the same coset. The fact that this is a homomorphism of groups comes from multiplicativity of the order of a meromorphic function. □

A divisor in the form div$f$ for a meromorphic function f on S is said to be *principal*. Two divisors D, D' on S are *linearly equivalent* if their difference is principal. The relation $D \sim D' \Leftrightarrow D - D' = \text{div}(f)$, for $f \in \mathcal{M}(S)$ is an equivalence relation on S: the quotient group $\text{Div}(S)/\sim$, is the *Picard group* of S, $\text{Pic}(S)$. It is isomorphic to a subgroup of the group of the line bundles on S modulo isomorphism, by using the injection that we are about to discuss.

Take $\{f_\alpha\}$ representing D as for proposition 2.3.7 and consider the quotients $g_{\alpha\beta} = f_\alpha/f_\beta$: being holomorphic, nowhere vanishing on intersections $U_\alpha \cap U_\beta$, and satisfying the cocycle conditions, they are the transition functions of a certain line bundle L. This bundle does not depend on the set of functions representing D; if $\{f'_\alpha\}$ is another such set then for each $p \in S$ it is $\text{ord}_p f_\alpha = \text{ord}_p f'_\alpha$ so that the functions $h_\alpha = f'_\alpha/f_\alpha$ are holomorphic and nowhere vanishing on U_α. Then setting $g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta}$ we see

$$g'_{\alpha\beta} = \frac{f'_\alpha}{f'_\beta} = g_{\alpha\beta}\frac{h_\alpha}{h_\beta}$$

Differing by a holomorphic function the transition functions $g_{\alpha\beta}$, $g'_{\alpha\beta}$ yield isomorphic vector bundles. Moreover if $\{f_\alpha\}$ and $\{f'_\alpha\}$ are two collections associated with divisors D and D' mapping respectively to bundles L and L' then $D + D'$ is represented by $\{f_\alpha f'_\alpha\}$ which maps to the line bundle $L \otimes L'$.

Finally this map respects the quotient structure of the two groups: if D is a divisor associated with a set of meromorphic functions $f_\alpha = g_\alpha/h_\alpha$, $g_\alpha, h_\alpha \in \mathcal{O}_{U_\alpha}$. On $U_\alpha \cap U_\beta$, $g_{\alpha\beta} = 1$ so L is the trivial bundle. Assume conversely that L is isomorphic to the trivial bundle; this means that $g_{\alpha\beta} = f_\alpha/f_\beta$ for generic holomorphic functions f_α, f_β non vanishing on the whole U_α and U_β; on the other hand D corresponding to L is given by a cocycle $\{h_\alpha\} \in H^0(S, \mathcal{M}^*/\mathcal{O}^*)$ and so we also have $g_{\alpha\beta} = h_\alpha/h_\beta$ whence, on intersections

$$\frac{f_\alpha}{f_\beta} = \frac{h_\alpha}{h_\beta} \Rightarrow \frac{h_\alpha}{f_\alpha} = \frac{h_\beta}{f_\beta}.$$

Consequently the representation $\{h_\alpha/f_\alpha\}$ reveals that D is the divisor of a meromorphic function.

We turn now our attention to the conditions needed for a line bundle L to be in the image of the above injection, in other words we ask when a line bundle can be thought as a (class of linear equivalence of a) divisor of S.

Definition 2.3.8. Let $\pi : L \to S$ be a line bundle on a Riemann surface S. A *meromorphic* section $s : S \to L$ is a holomorphic function s on an open subset $U = S \setminus \{p_1, \ldots, p_n\}$ of S, with $p_1, \ldots, p_n \in S$, for which $\pi \circ s = id_U$, and such that it is a meromorphic function on each local trivialization $\{\phi_\alpha, U_\alpha\}$ of L, that is, such that $pr_2 \circ \phi_\alpha \circ s : U_\alpha \to \mathbb{C}$ is a meromorphic function for every α.

A meromorphic section s of S restricted to a trivialization $\{U_\alpha, \phi_\alpha\}$ yields meromorphic functions $s_\alpha = pr_2 \circ \phi_\alpha \circ s$ such that, if $g_{\alpha\beta}$ are transition functions on L, $s_\alpha / s_\beta = g_{\alpha\beta}$: hence $\{s_\alpha\} \in H^0(S, \mathcal{M}^*/\mathcal{O}^*)$. Conversely given such a collection consider, wherever defined on U_α the functions $\tilde{s}_\alpha : U_\alpha \to U_\alpha \times \mathbb{C}$ given by $\tilde{s}_\alpha(p) = (p, s_\alpha(p))$. These functions are such that $\phi_\alpha^{-1} \circ \tilde{s}_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ is a meromorphic section of L on U_α; furthermore $s_\alpha = \phi_\alpha \circ \phi_\beta^{-1} \circ s_\beta$ on $U_\alpha \cap U_\beta$ implies $\tilde{s}_\alpha = \tilde{s}_\beta$ so that by the sheaf axiom a unique global meromorphic section s on S coming from $\{s_\alpha\}$ exist.

Suppose now we are given two global meromorphic sections $s = \{s_\alpha\}$, $s' = \{s'_\alpha\}$ of L having transition functions $g_{\alpha\beta}$. Then, on $U_\alpha \cap U_\beta$

$$\frac{s_\alpha}{s'_\alpha} = \frac{g_{\alpha\beta} s_\beta}{g_{\alpha\beta} s'_\beta} = \frac{s_\beta}{s'_\beta}$$

so the quotient if two meromorphic sections is the divisor of a globally defined meromorphic function, thus s and s' represent the same linear equivalence class of divisors. In conclusion to the set of meromorphic sections of a bundle L corresponds a whole linear equivalence class of divisors: we have proved

Proposition 2.3.9. *If a line bundle L has a global nontrivial holomorphic section s then there is a divisor D associated with L, whose linear equivalence class is uniquely determined by any of these sections.*

Suppose now we are given the canonical bundle $\Lambda^1(T^*S)$ on S, whose associated sheaf of holomorphic sections is Ω^1 the sheaf of holomorphic 1-forms. Since any Riemann surface has a nonconstant globally defined meromorphic function f, the form $f dz$ is a global meromorphic section of $\Lambda^1(T^*S)$; therefore the *canonical divisor class* K associated with the canonical sheaf $\mathcal{K} = \Omega^1$ exists on a Riemann surface.

In the group structure of invertible sheaves the inverse of a sheaf $\mathcal{V}$ on S is the sheaf $\mathcal{V}^*$; take an equivalence class of divisors D and associate with it the sheaf of sections $\mathcal{O}(D)$ of the bundle defined by D; by theorem 2.2.6 (in the compact case) $H^1(S, \mathcal{O}(D)) \cong H^0(S, \Omega^1(\mathcal{O}(D)^*))$; now $\Omega^1(\mathcal{O}(D)^*) = \Omega^1 \otimes \mathcal{O}(D)^*$ is the sheaf of sections of the line bundle corresponding to the divisor $K - D$ so in this situation the Theorem reads

Theorem 2.3.10. *Let S be a compact connected Riemann surface, D an equivalence class of divisors on M. Then*

$$H^q(S, \mathcal{O}(D)) \cong H^{1-q}(S, \mathcal{O}(K - D))$$

for $q = 0, 1$.

Clearly, the fact that S is a Riemann surface is only a sufficient condition. This result is valid for any n-dimensional complex manifold whose canonical divisor is defined; remarkably this happens when S is an algebraic variety. The case $q = 0$, $D = 0$ has an intepretation in terms of geometric invariants on S: the dimension of $H^0(S, \Omega^1)$ is called the *geometric genus* ρ_g of the surface while $\dim H^1(S, \mathcal{O})$ is the *arithmetic genus* ρ_a. The duality then estabilishes that these two are equal on Riemann surfaces, thus it makes sense to talk generically about *the genus* g of S.

2.3.5 The Riemann-Roch Theorem for Riemann Surfaces

By using Serre Duality an easy proof of Riemann and Roch Theorem is at hand. This theorem links the genus g of a Riemann surface S, the degree of a divisor D and the dimensions of the cohomology groups of D. The *degree* of a divisor $D = \sum_p n_p p$ is the integer $deg(D) = \sum_p n_p$. Set $h^0(D) = \dim H^0(S, \mathcal{O}(D)$ and $h^1(D) = \dim H^1(S, \mathcal{O}(D))$, We have the following proposition

Proposition 2.3.11. *The integer $h^0(D) - h^1(D) - deg(D)$ does not depend on D and equals $1 - g$.*

Proof. It is sufficient to show that the sum in the hypotheses is equal to

$$h^0(D + P) - h^1(D + P) - deg(D + P)$$

for every point $P \in S$. Consider the sheaf $\mathcal{O}(D)$ as a sheaf of sections of the vector bundle defined by D. It is a subbundle of $\mathcal{O}(D + P)$ because every section of D is a section of $D + P$; in order to verify this we give a more explicit description of the vector spaces $\Gamma(U, \mathcal{O}(D))$.

Let D be a divisor and choose a trivialization $\{U_\alpha\}$ of its line bundle. Consider a nonzero meromorphic function s on $U \subset S$ such that $\mathrm{div} s \geq -D$ and take $\{g_\alpha\}$ defining D over $\{U_\alpha\}$. The functions $f_\alpha = g_\alpha s$ are such that, for $p \in U$, $\mathrm{div}(f_\alpha) \geq 0$ and on intersections $f_\alpha = g_{\alpha\beta} f_\beta$, where $g_{\alpha\beta} = g_\alpha / g_\beta$ are the transition functions of D. We then see that the meromorphic function s determines a section $\{f_\alpha\}$ of $\mathcal{O}(D)$ on U that is easily seen not depending on the choosen $\{g_\alpha\}$. Conversely if $\{s_\alpha\} \in \Gamma(U, \mathcal{O}(D))$ then $f_\alpha = s_\alpha / g_\alpha$ are such that

$$\frac{s_\alpha}{g_\alpha} = \frac{g_{\alpha\beta} s_\beta}{g_{\alpha\beta} g_\beta} = \frac{s_\beta}{g_\beta}$$

so the f_αs define a meromorphic function f on U, and $\mathrm{ord}_p(f_\alpha)$+$\mathrm{ord}_p(g_\alpha) \geq 0$ because s_α is holomorphic: this implies $\mathrm{div}(f) \geq -D$. Therefore the sections of $\mathcal{O}(D)$ on any open set U can be identified with the vector space

$$\{f \in \mathcal{M}^*(U), \, \mathrm{div}(f) \geq -D\}$$

which is finite dimensional; this correspondance is also an isomorphism of vector spaces.

From this we easily see that if $f \in \Gamma(U, \mathcal{O}(D))$ then $f \in \Gamma(U, \mathcal{O}(D+P))$ which yields a sheaf inclusion $\mathcal{O}(D) \hookrightarrow \mathcal{O}(D+P)$. We want now to describe the quotient sheaf $\mathcal{Q} = \mathcal{O}(D+P)/\mathcal{O}(D)$.

On every $x \in S$ each meromorphic function f with $\text{div}(f) \geq -D-P$ has germ f_x given by its Laurent series; if $x \neq P$ then f on a neighbourhood of x not containing P has the same power series expansion of a function whose divisor is $\geq -D$ so $\mathcal{Q}_x = \mathcal{O}_x(D+P)/\mathcal{O}_x(D) = 0$. Instead around P we can write f as

$$f(z) = \frac{\text{Res}_P(f)}{(z-P)} + \sum_{k\geq 0} \frac{f^{(k)}(P)}{k!}(z-P)^k$$

so the coset of f_x modulo $\mathcal{O}_x(D)$ is $\text{Res}_P f/(z-P)$. By taking the residue the vector space of all these local expressions is seen to be isomorphic to $\mathbb{C}$, whence $\mathcal{Q}_P \cong \mathbb{C}$. The sheaf $\mathcal{Q}$ having these stalks is the so called *skyscraper sheaf* concentrated on P and is defined on sections by $\mathcal{Q}(U) = \mathbb{C}$ if $P \in U$, 0 otherwise. Obviously $H^0(S, \mathcal{Q}) = \mathbb{C}$ and $H^q(S, \mathcal{Q}) = 0$ if $q > 0$. Thus the exact sequence of sheaves

$$0 \to \mathcal{O}(D) \to \mathcal{O}(D+P) \to \mathcal{Q} \to 0$$

induces a long exact cohomology sequence

$$\begin{aligned} 0 \to H^0(S, \mathcal{O}(D)) \to H^0(S, \mathcal{O}(D+P)) \to \mathbb{C} \to H^1(S, \mathcal{O}(D)) \to \\ \to H^1(S, \mathcal{O}(D+P)) \to 0 \end{aligned} \tag{2.27}$$

If finite, the alternating sum of the the dimensions of the vector spaces of a long exact sequence equals 0, hence from 2.27 we have the equation

$$h^0(D) - h^1(D) + 1 - h^0(D+P) + h^1(D+P) = 0$$

but evidently $\deg(D+P) = \deg(D)+1$ from which

$$h^0(D) - h^1(D) - deg(D) = h^0(D+P) - h^1(D+P) - deg(D+P)$$

that is the first claim. For $D = 0$, the sheaf $\mathcal{O}(D)$ is the sheaf of holomorphic functions on S so $h^0(0) = 1$, $\deg(D) = 0$ and $h^1(0) = g$ so that the second claim follows. □

Corollary 2.3.12 (Riemann-Roch Theorem). *For every divisor D on a Riemann surface S of genus g*

$$h^0(D) - h^0(K-D) = deg(d) + 1 - g$$

Proof. Proposition 2.3.11 and theorem 2.3.10. □

We can go now further on and identify the genus g of a surface with the *topological genus* of the surface.

The topological genus of a Riemann Surface is defined as half of the number of the sides of the regular polyhedron whose topological quotient defines S, and coincides with the half of the first Betti number of S, that is, half of the dimension of the first singular homology group of S. We indicate the topolological genus of S with g_{top} while g is the genus previously defined. It is a basic fact that a Čech cohomology group of a constant sheaf is isomorphic to the respective singular homology group. For an orientable Riemann surfaces it is

$$\begin{aligned} H^0(S,\mathbb{C}) &\cong \mathbb{C} \\ H^1(S,\mathbb{C}) &\cong \mathbb{C}^{2g_{top}} \\ H^2(S,\mathbb{C}) &\cong \mathbb{C}. \end{aligned}$$

Consider the exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}_S \xrightarrow{d} \Omega^1_S \to 0$$

and the associated cohomology sequence

$$\begin{aligned} 0 \to H^0(S,\mathbb{C}) \to H^0(S,\mathcal{O}_S) \to H^0(S,\Omega^1_S) \to H^1(S,\mathbb{C}) \to \\ \to H^1(S,\mathcal{O}_S) \to H^1(S,\Omega^1_S) \to H^2(S,\mathbb{C}) \to 0. \end{aligned} \tag{2.28}$$

It is

$$\dim H^0(S,\mathcal{O}_S) = \dim H^0(S,\mathbb{C}) = \dim H^2(S,\mathbb{C}) = \dim H^1(S,\Omega^1_S) = 1$$

and

$$\dim H^0(S,\Omega^1_S) = \dim H^1(S,\Omega^1_S) = g$$

whence, being 0 the alternating sum of dimensions

$$1 - 1 + g - 2g_{top} + g - 1 + 1 = 0$$

so that finally $g = g_{top}$.

Remark. Serre duality does assure that proposition 2.3.11 and corollary 2.3.12 are equivalent, so one may ask why not being happy with the first formulation. The fact is that the original statement is in the latter form and has an interpretation in terms of *linear systems of divisors* that cannot be recovered from the form 2.3.11. A *complete linear system* $|D|$ of a divisor D is the set of all effective divisors linearly equivalent to D. This set, modulo multiplication for global never vanishing holomorphic function (i.e. a complex number), is in $1-1$ correspondence with the set of all nonzero meromorphic functions whose divisor is $\geq -D$, which by the identification given in the proof of proposition 2.3.11 is nothing but $H^0(S,\mathcal{O}(D))$. Therefore $|D|$ has a natural structure of projective space of dimension $h^0(D)-1$. Classically the number $h^0(D)$ is denoted $\ell(D)$.

Chapter 3

The Theorem in the Algebraic Setting

The Duality Theorem for coherent sheaves on algebraic varieties appeared in [23] in 1954 almost at the same time of that for analytic manifolds. Serre's proof was bound to the language of classic algebraic geometry since by that time scheme theory had yet to be established by Grothendieck. Our proof is taken from [14] and uses schemes; it is limited to projective spaces, but a more general statement for projective schemes could have been proved with a little more work, yet remaining within the limits of this exposition.

The main difference between the coherent case and the locally free one is that the dual of the cohomology group $H^q(X, \mathcal{F})$ will not be another cohomology group, but the Ext^q group. The appearing of a right derived functor clearly implies the assumption of a more abstract homological viewpoint. The first section is a quick review of some basics of scheme theory. Section 3.2 introduces coherent sheaves of modules; section 3.3 illustrates the construction of the Proj of an algebra; section 3.4 deals with some of the required aspects of homological algebra; finally 3.5 is dedicated to the proof of the Theorem.

3.1 Generalities on Schemes

Algebraic geometry is classically interested in the study of geometrical structures that locally is represented as the set of zeros of algebraic equations. To any affine open set of such an algebraic (quasiprojective) variety one can associate in an invariant way a certain algebra $k[X]$ of quotients of a polynomial ring with coefficients over an algebraically closed field k, which is a particular kind of k-algebra. This algebra represents the algebra of the regular functions and its maximal spectrum gives locally the points of the variety. One may ask what happens if we want to change the point of view and define instead a topological object by means of a well-behaved commutative ring A that has to play the same role of $k[X]$.

This questions leads to the definition of affine scheme; the difference is that in order to consitently define a notion of *morphism* between affine schemes we have to consider the whole prime spectrum Spec$A$ of A. General schemes are topological spaces locally isomorphic to affine schemes. Scheme theory is the modern field of interest and research of algebraic geometry, since in its language many geometrical problems may be more easily related to various different aspects of mathematics, as number theory, mathematical physics, and so on.

We start from describing the construction of a scheme along with its main properties. The spectrum Spec$A$ of a commutative unitary ring A is the set of prime ideals of A. This set can be given the Zariski topology by considering, for $I \subset A$, the class of closed sets $V(I) = \{P \in \text{Spec}A | I \subset P\}$. The complementary of $V(I)$ is the open set called $D(I)$. A family of open sets generating this topology is $\{D(f), f \in A\}$ consisting of the *principal open sets* $D(f)$. We define continuous functions between ring spectrums by the only possible choice: if $\phi : A \to B$ then ${}^a\phi : \text{Spec}B \to \text{Spec}A$ sending a prime P to its inverse image $\phi^{-1}(P)$ is a continuous function because $({}^a\phi)^{-1} : V(E) = V(\phi(E))$ if $E \subset A$.

In particular one sees by the correspondence of ideals containing a fixed ideal of a ring, and the ideals of the quotient of that ring for the fixed ideal, that $V(I)$ is homeomorphic to SpecA/I. Similarly a principal open set $D(f)$ is homeomorphic to the spectrum of the localizations A_f in the multiplicative set $1, f, f^2, \ldots, f^n, \ldots$, for f not a nilpotent element.

A remarkable fact is that in most cases on Spec$A$ there are not only nonclosed points, but also dense ones. Suppose that the set of zero-divisor (the nilradical ideal N) is prime in A; then it is contained in every prime ideal and $V(N)$ is the whole Spec$A$, whence N is a dense point, called a *generic point*. In particular if A is an integral domain (0) is a generic point. The existence of generic points is strictly connected to topological *irreducibility*; Spec$A$ cannot be written as a union of proper closed sets if and only if a generic point exists.

It is possible to understand elements of A as 'functions' having values for each $P \in A$ in different residue fields A/P by taking their images in the quotient. The next step is to construct, starting from A, a sheaf of rings, the *structure sheaf* $\mathcal{O}$, that has the same meaning of the sheaf of regular functions on a variety, and whose set of global sections coincides with the whole A. According to the fact that, for $f \in A$, the spectrum of the localizations A_f of A at the multiplicative part $\{f^n | n \geq 1\}$ is homeomorphic to the principal open set $D(f)$, we define, for every $f \in A$, the sections $\mathcal{O}(D(f)) := A_f$. For every other subset U

$$\mathcal{O}(U) = \lim_{D(f) \subset U} \mathcal{O}(D(f))$$

where the limit indicates the projective limit over every principal subset contained in U. Restrictions maps exist by intrinsic properties of ring localizations and can be carried over to the limit; this presheaf is in fact a sheaf and one can check that $\mathcal{O}(\mathrm{Spec}A) = A$. The stalks $\mathcal{O}_x$ at $x \in \mathrm{Spec}$ of this sheaf have a similar interpretation of that of the stalks of holomorphic functions on a complex manifold, or of that of local ring of regular functions on an algebraic variety. If A_x is the localization of the ring A at the prime x then $\mathcal{O}_x = A_x$. The pair $(\mathrm{Spec}A, \mathcal{O}_A)$ is our *affine scheme*.

Given a continuous function $f : X \to Y$ and a sheaf $\mathcal{F}$ on X one can consider the direct image sheaf $f_*\mathcal{F}$ on Y by taking $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$ for an open set $U \subset Y$. Restrictions are defined in the same way.

A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ a sheaf of rings over X. A morphism of a ringed space $(X, \mathcal{O}_X)$ to a ringed space $(Y, \mathcal{O}_Y)$ is a pair $(\phi, \phi^\#)$ where $\phi : X \to Y$ is a continuous map and $\phi^\# : \mathcal{O}_Y \to \phi_*\mathcal{O}_X$ a morphism of sheaves on Y. An isomorphism of ringed space is defined in the natural way.

Definition 3.1.1. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ such that there exist an open covering $\{U_\alpha\}$ of X and a set of commutative unitary rings A_α such that $(U_\alpha, \mathcal{O}|_{U_\alpha})$ is isomorphic, as a ringed space, to $(\mathrm{Spec}A_\alpha, \mathcal{O}_{A_\alpha})$, for all α. If such a covering can be taken finite and such that A_α are all Noetherian the scheme is then called *Noetherian*.

Whenever no confusion may arise the subscript from the structure sheaf is omitted. By *dimension* of a scheme X we mean simply its dimension as a topological space, that is, the maximum integer n for which a chain of irreducible distinct closed subsets

$$Z_0 \subset Z_1 \subset \ldots \subset Z_n$$

exists. If X is affine this is just the Krull dimension of the ring defining it.

A morphism of schemes is quite more complicated than a morphism of ringed spaces; the reason is the fact that if we defined a morphism of schemes to be simply a morphism of ringed spaces then the vanishing of a function on the image of a point x does not imply that the pullback of that function vanishes on x; we want this to happen. However actually true if instead we consider to *local* ringed space morphisms, that is morphisms $(\phi, \phi^\#)$ of ringed spaces such that their restriction on affine subsets $\mathrm{Spec}A \subset X$, $\mathrm{Spec}B \subset Y$ inducing on stalks a local ring homomorphism $\phi_x : B_{\phi(x)} \to A_x$, that is to say, ϕ_x is an homomorphism such that the inverse-image of the maximal ideal of A_x is that of $B_{\phi(x)}$.

Let X and Y be two schemes. A *closed immersion* is a morphism of schemes $i : X \to Y$ inducing a homeomorphism of the (underlying topological space of) X with a closed subset of Y and such that the morphism of sheaves $i^\# : \mathcal{O}_Y \to i_*\mathcal{O}_X$ is surjective. The kernel of the morphism $i^\#$ is a sheaf called the *ideal sheaf* of Y. As a specific example for an ideal I of A we can consider the homomorphism of $\pi : A \to A/I$ and the homeomorphism between $V(I)$ and $\mathrm{Spec}A/I$. Commuting localization and quotient, the corresponding morphism $\pi^\#$ of sheaves is surjective on stalks. We deduce that a closed subset $Y \subset \mathrm{Spec}A$ has as many subscheme structures as the ideals I such that $V(I) = Y$.

3.2 Sheaves of Modules. Coherent Sheaves

The formalism of $\mathcal{O}$-modules has been already used implicitly in the analytic category. For completeness and to clarify what follows we now make it explicit in the algebraic context. Let $(X, \mathcal{O})$ be a scheme. A sheaf $\mathcal{F}$ of abelian groups over X is called an *$\mathcal{O}$-module* (or simply a module) over X if $\mathcal{F}(U)$ has a structure of $\mathcal{O}(U)$-module for every open set $U \subset X$, in a way that the restricitons are compatible with the exterior product. A morphism of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ is a sheaf morphism such that on open sets is a homomorphism of $\mathcal{O}(U)$-modules. Subsheaves, kernel, cokernel, tensor product modules, are defined as for sheaves. The stalks of a sheaf of modules are modules over the corresponding stalks of the structure sheaf. Given a pair of modules $\mathcal{F}$ and $\mathcal{G}$ over X one can form the additive group $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})$ of morphisms of $\mathcal{O}$-modules between $\mathcal{F}$ and $\mathcal{G}$. Very important is also the $\mathcal{H}om$ sheaf given by the functor

$$U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where the restricted sheaf $\mathcal{F}|_U$ is defined by $\mathcal{F}(V)$ if $V \subset U$, 0 otherwise. A *sheaf of ideals* is a sheaf of $\mathcal{O}$-modules which is a subsheaf of $\mathcal{O}$ (its sections are indeed ideals).

For a morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and a pair $\mathcal{F}, \mathcal{G}$ of $\mathcal{O}$-modules over them, the direct image sheaf $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module. The sheaf $f^{-1}\mathcal{G}$ on X is defined by letting correspond to $U \subset X$ the inverse limit $\lim_{V \supset f(U)} \mathcal{G}(V)$; this is a sheaf of $f^{-1}\mathcal{O}_Y$-modules. Of more interest is the *inverse image* sheaf $f^*\mathcal{G}$: it is the $\mathcal{O}_X$-module given by the tensor product

$$f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

Peraphs the most important class of sheaves on a scheme X are the *coherent* sheaves of $\mathcal{O}_X$-modules, i.e. those sheaves of modules that locally can be obtained from a certain module M on a ring A by means of the following construction. Let $\mathrm{Spec}A$ be an affine scheme and M an A-module. Define for $f \in A$ the *localization* of M at f to be the A-module $M_f = M \otimes_A A_f$. We assign M_f to any principal open set $D(f)$; by localization properties for $g \in A$ such that $D(g) \subset D(f)$ a morphism of A-modules $M_f \to M_g$ exists. For an arbitrary open set $U \subset \mathrm{Spec}A$ we let $\tilde{M}(U)$ be the inverse limit of M_f on all principal open sets contained in U. We have constructed a sheaf of $\mathcal{O}$-modules to be called $\tilde{M}$, whose stalks are at each prime p the localizations M_p and whose global sections are the module M. This construction evidently coincides with that of $\mathcal{O}$ if $M = A$.

Definition 3.2.1. A sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on a scheme X is *quasi-coherent* if there exists an open affine covering $\{U_\alpha\}$ such that $\mathcal{F}|_{U_\alpha}$ is of the form $\tilde{M}_\alpha$ for a certain A_α-module M_α, with $\mathrm{Spec}A_\alpha \cong U_\alpha$. It is *coherent* if every M_α can be choosen finitely generated.

The kernel, cokernel, image sheaf of a morphism $\mathcal{F} \to \mathcal{G}$ of quasi-coherent sheaves on a scheme X are quasi-coherent. In general this is false for coherent sheaves, unless we take X Noetherian. It turns out that if we want - and in fact we do - the class $Coh(X)$ of coherent sheaves to be an abelian category, we have to make this reasonable hypothesis on X. In the following sections, if not explicity mentioned, this will always be assumed.

Let $X =$Spec$A$; the functor $M \to \tilde{M}$ is an exact functor between the category of A-modules to that of $\mathcal{O}_X$-modules which respects tensor products and direct sums. In particular the tensor product and direct sum of coherent sheaves are coherent. If X is Noetherian the same holds for finitely generated A-modules and coherent sheaves.

For a scheme morphism $f : X \to Y$, with X, Y Noetherian, and a coherent sheaf $\mathcal{F}$ on Y, the inverse image sheaf $f^*\mathcal{F}$ is always coherent. If $\mathcal{G}$ is coherent on X the direct image $f_*\mathcal{G}$ is noetherian if f is a closed immersion. Therefore if Y is a closed subscheme of a Noetherian scheme X its ideal sheaf $\mathcal{I}$ is coherent. Conversely for any ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ of X there is a unique closed subscheme Y such that the ideal sheaf of the immersion $i : Y \hookrightarrow X$ is $\mathcal{I}$; this immersion makes $\mathcal{O}_Y$ isomorphic to the quotient sheaf $i_*\mathcal{O}_X/\mathcal{I}$ and yields the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

In particular we see that a locally free sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on X is a coherent sheaf since is locally in the form $\tilde{M}$ for a finite sum $M_\alpha = \bigoplus A_\alpha$ over an affine covering (Spec$A_\alpha, \mathcal{O}_{A_\alpha}$) of X. If this sum has rank 1 on any open set then $\mathcal{F}$ is *invertible*.

The dual sheaf of a locally free sheaf is the sheaf $\mathcal{F}^* = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$. We have an isomorphism $(\mathcal{F}^*)^* \cong \mathcal{F}$; furthermore if $\mathcal{F}$ is locally free and $\mathcal{E}, \mathcal{G}$ are any $\mathcal{O}$-modules there are isomorphisms

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong \mathcal{F}^* \otimes \mathcal{G} \tag{3.1}$$

$$\text{Hom}(\mathcal{F} \otimes \mathcal{E}, \mathcal{G}) \cong \text{Hom}(\mathcal{E}, \mathcal{H}om(\mathcal{F}, \mathcal{G})). \tag{3.2}$$

3.3 Proj of an Algebra and Twisted Sheaves

We analyze now an important scheme that is the abstract generalization of the projective variety $\mathbb{P}^n$, the Proj of an algebra. Let $S = \bigoplus_{k\geq 0} S_k$ be a graded ring and S_+ the ideal $\bigoplus_{k>0} S_k$. Let Proj$S$ be the set of all homogeneous prime ideals not containing S_+. As we did in the case of affine schemes we give a topology on Proj$S$ defining $V(I) = \{P \in \text{Proj}S | \, I \subset P\}$, for every subset I of S, to be the class of closed sets of Proj$S$.

It is possible to define a sheaf of rings $\mathcal{O}$ over Proj$S$, in the following manner. In the localization S_p at a prime homogeneous ideal p consider $S_{(p)}$, the subring of elements of degree 0 of the graded ring S_p. We let $\mathcal{O}(U)$ be the set of functions $s : U \to \amalg_{p\in U} S_{(p)}$ such that $s(p)$ is in $S_{(p)}$ for every $p \in U$, and such that s is locally a quotient, which means that for every p

in U there exists an open neighbourhood V of p such that for each $q \in V$ there are $f \in S \setminus q$ and $a \in S$ of same degree such that $s(q) = a/f$. $\mathcal{O}$ is a presheaf which is easily seen to be a sheaf of rings making Proj$S$ into a scheme. More precisely the following proposition holds:

Proposition 3.3.1. *Let S be a graded ring. Consider the pair $(ProjS, \mathcal{O})$.*

(a) For each $p \in ProjS$ it is $\mathcal{O}_p \cong S_{(p)}$

(b) The family of open sets $D_+(f) = V(f)^c$, f varying in S_+, forms an open affine covering of $ProjS$. Moreover

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong (SpecS_{(f)}, \mathcal{O})$$

where $S_{(f)}$ here indicates the subring of elements of degree 0 in the ring of fractions obtained by the multplicative part $\{f^n | n \geq 0\}$.

Proof. [14] chap. II p. 76 □

The most important case, and the only one that will occurr in this dissertation, is when $S = A[X_0, \ldots, X_n]$ for a certain Noetherian ring A; in this case we denote Proj$S = \mathbb{P}^n_A$ which is called the *projective n-space over A*. In the case $A = k$ is an algebraically closed field $\mathbb{P}^n_k$ is a scheme whose set of closed point is isomorphic to the projective space over k meant as an algebraic variety. Of course $\mathbb{P}^n_A$ is a Noetherian scheme

We now define the twisting sheaves. On Proj$S$ can be carried on a construction of a sheaf of $\mathcal{O}$-modules parallel to that of $\tilde{M}$ on Spec$A$, for an A-module M. Let S be a graded ring and M a graded module over S. In analogy with the structure sheaf of Proj$S$ define $\tilde{M}(U)$ to be the set of functions from U in $\amalg_p M_{(p)}$ such that each p is sent in $M_{(p)}$, and that are locally fractions. $\tilde{M}$ has the following properties.

Proposition 3.3.2. *Let S and M as above. The sheaf $\tilde{M}$ is such that*

(a) The stalks $\tilde{M}_p$ are isomorphic to the ring $M_{(p)}$ of elements of degree 0 in the localization at p;

(b) For any principal open set $D(f)$, $f \in S_+$, it is $\tilde{M}|_{D(f)} \cong \widetilde{M_{(f)}}$;

(c) $\tilde{M}$ is quasi-coherent. If S is Noetherian and M finitely generated then $\tilde{M}$ is coherent.

Proof. [14] chap. II.2 p. 76. □

As for affine schemes, the rule associating to a Noetherian S-algebra M the coherent sheaf $\tilde{M}$ is an exact functor between the category of S-algebras to that of coherent sheaves on Proj$S$.

The most relevant sheaves on Proj$S$ are the twisting sheaves $\mathcal{O}(n)$. Let S be a graded algebra with the standard positive grading, $S = \bigoplus_{k \geq 0} S_k$. We define the shifted algebra $S(n)$

to be S with a different grading, namely $S(n)_k = S_{n+k}$, or equivalently $S(n) = \bigoplus_{k\geq 0} S_{k-n}$: we then set $\mathcal{O}(n) = \tilde{S(n)}$. the sheaf $\mathcal{O}(0)$ is just the structure sheaf. $\mathcal{O}(1)$ is called the *Serre twisted sheaf.* For any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on Proj$S$ the *twisted sheaf* $\mathcal{F}(n)$ is the tensor product of modules $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(n)$. The following proposition illustrates the nature of $\mathcal{O}(n)$

Proposition 3.3.3. *$\mathcal{O}(n)$ is an invertible sheaf on ProjS; if ProjS $= \mathbb{P}^n_k$ for an algebraically closed field k then any invertible sheaf is in the form $\mathcal{O}(r)$, $r \in \mathbb{Z}$. Moreover for any couple of integers n, m we have $\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m)$.*

Proof. [14] chap. II par.5 p. 117. □

By 'twisting' coherent sheaves over $\mathbb{P}^n_A$ a sufficient number of times one can describe the resulting sheaf in the form of a quotient as follows. Let X be a scheme: we say that an $\mathcal{O}$-module $\mathcal{F}$ is *generated by its global sections* if there exists a set $\{s^i\}$ of global sections such that for every $x \in X$ the stalk $\mathcal{F}_x$ is generated as an $\mathcal{O}_x$-module by the germs $\{s^i_x\}$.

Theorem 3.3.4 (Serre). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_A$. Then there exists a number n_0 such that for all $n \geq n_0$ the twisted sheaf $\mathcal{F}(n)$ is generated by a finite number of global sections.*

Proof. [14] chap 2. p. 121. □

This theorem has a corollary that will be very useful:

Corollary 3.3.5. *A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_A$ is the a quotient of a finite direct sum sheaf $\mathcal{E} = \bigoplus_{i=1}^m \mathcal{O}(-n)$ for all $n \geq n_0$ (n_0 depending on $\mathcal{F}$).*

Proof. Let n_0 be as for 3.3.4 with sections $s_i, \ldots, s_m$ generating $\mathcal{F}(n)$, $n \geq n_0$. A surjection $\bigoplus_{i=0}^m \mathcal{O} \to \mathcal{F}(n)$ is established by sending each m-uple $(f_i)_i$ of sections f_i on an open set U in $\sum_i f_i s_i|_U$. By right exactness of tensoring we obtain a surjection $\mathcal{E} = \bigoplus_{i=1}^m \mathcal{O}(-n) \to \mathcal{F}$ whose cokernel is the desired quotient. □

3.4 Derived Functors and Cohomology

The aim of this section is to introduce δ-functors, right derived functors and their universal properties. As usual we are assuming familiarity with the basic facts on categories and functors.

A category consists of a class of objects $\mathcal{C}$ and a set of morphisms Hom(A, B) for every pair of objects A and B in $\mathcal{C}$. A morphism f between two objects is a *monomorphism* whenever for any pair of morphisms (g, h) from a third object such that $f \circ g = f \circ h$ then $g = h$. f is an *epimorphism* if for any pair of morphisms (g, h) to a third object is such that $g \circ f = h \circ f$ implies $g = h$. A category is *additive* if a sum of any two objects is defined and every Hom-set contains a 0 object.

Definition 3.4.1. An *abelian category* $\mathcal{C}$ is an additive category containing finite direct sums of any objects, such that for each pair of objects A, B and for each $m \in \text{Hom}(A, B)$, m has both a kernel and a cokernel; every monomorphism is the kernel of its cokernel; every epimorphism is the cokernel of its kernel and finally any morphism can be factored into an epimorphism followed by a monomorphism.

From now on $\mathcal{C}$ will always be abelian.

A cochain complex in an abelian category $\mathcal{C}$ is of course a sequence $A^* = (A^i, d_i)$ of objects and morphisms d_i from A^i to A^{i+1} such that $d_{i+1} \circ d_i = 0$ so that cohomology objects $h^i(A^*) = \text{ker}d_i/\text{im}d_{i-1}$ are defined. If $h^i(A^*) = 0$ for all i then A^* is exact. A chain complex is the dual sequence of a cochain complex.

An *injective* object I is an object in $\mathcal{C}$ such that for all morphims $f : A \to I$ and all monomorphisms $i : A \hookrightarrow B$ exists a morphism $g : B \to I$ such that $f = g \circ i$. An object P is instead *projective* if for any morphism $g : P \to A$ and epimorphism $\pi : B \to A$ there exists a morphism $h : P \to B$ such that $g = \pi \circ h$. A more frequently used characterization of injective objects is that I is injective if and only if the functor $\text{Hom}(\bullet, I)$ is exact.

For every object A in $\mathcal{C}$ an *injective resolution* of A is a sequence (I^i, d_i) of injective objects and morphism $d_i : I^i \to I^{i+1}$ together with a monomorphism $\epsilon : A \to I^0$ such that

$$0 \to A \xrightarrow{\epsilon} I^0 \xrightarrow{d_0} I^1 \to \cdots$$

is exact. A *projective* resolution of an object B is a sequence (P_i, d_i) of projective objects and morphism $d_i : P_{i+1} \to P_i$ together with an epimorphism $\pi : P_0 \to B$ such that

$$\cdots \to P_1 \xrightarrow{d_0} P_0 \xrightarrow{\pi} B \to 0.$$

is exact.

A category $\mathcal{C}$ has *enough injectives* if for all objects A in $\mathcal{C}$, $\exists f : A \to I$ where I is injective; has *enough projectives* if for all B in $\mathcal{C}$, $\exists g : P \to B$ where P is projective. It can be checked that in a category with enough injectives (projectives), injective (projective) resolutions of any object exist.

Ther following abelian categories will be used:

- The category *Ab* of abelian groups;
- The category *CochAb* of cochains of abelian groups;
- The category $Mod(X)$ of $\mathcal{O}$-modules over a scheme X;
- The categroy $Coh(X)$ of coherent sheaves of $\mathcal{O}$-modules over a Noetherian scheme X.

The categories *Ab* and *CochAb* have both enough injectives and projectives; the categories $Mod(X)$ and $Coh(X)$ have enough injective but not enough projectives.

Now we pass to functors. A functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories is called *additive* if it respects direct sums. It is *left exact* if $0 \to F(A) \to F(A')$ is exact for any exact sequence $0 \to A \to A'$, *right exact* if $F(A) \to F(A') \to 0$ is exact whenever $A \to A' \to 0$ is exact. It is *exact* if it is both left and right exact on short exact sequences $0 \to A \to A' \to A'' \to 0$.

Definition 3.4.2. Let $\mathcal{C}$ and $\mathcal{B}$ be abelian categories with enough injectives and $F : \mathcal{C} \to \mathcal{B}$ an additive covariant left exact functor. For any object A in $\mathcal{C}$ fix an injective resolution I^*; the *i-th right derived functor* of F is the covariant functor $R^iF : \mathcal{C} \to \mathcal{B}$ given by $R^iF(A) = h^i(F(I^*))$.

R^iF does not depend, up to an isomorphism, on the choosen I^*, because if two different injective resolutions of the same object are homotopy equivalent, so are their images under F, which therefore have isomorphic cohomologies. The properties of the right derived functors are the following:

Proposition 3.4.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor and R^iF its derived functors. Then*

1. *R^iF is additive for all i;*
2. *$F \cong R^0F$;*
3. *$R^iF(I) = 0$ for all injective objects I;*
4. *For any exact sequence*
$$0 \to A \to A' \to A'' \to 0$$
in $\mathcal{A}$ there exists a family of natural morphism $\delta^i : R^iF(A'') \to R^{i+1}F(A)$ inducing a long exact sequence
$$\cdots \to R^iF(A) \to R^iF(A') \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A) \to R^{i+1}F(A') \to \cdots$$
in $\mathcal{B}$.

Mutatis mutandis one can define the covariant *left derived* functors of a right exact functor from a category with enough projectives, and contravariant right (left) derived functors of a contravariant left (right) exact functor F from a category with enough injectives (projectives).

The right derived functors serve as a model for the following definition

Definition 3.4.4. A *covariant δ-functor* $\{F^i\}_{i\geq 0}$ is a collection of functors F^i between two abelian categories $\mathcal{A}$ and $\mathcal{B}$ such that for any exact sequence

$$0 \to A \to A' \to A'' \to 0$$

in $\mathcal{A}$ the induced sequence

$$\cdots \to F^i(A) \to F^i(A') \to F^i(A'') \xrightarrow{\delta^i} F^{i+1}(A) \to F^{i+1}(A') \to \cdots$$

is exact in $\mathcal{B}$ via certain *natural* morphisms $\delta^i : F^i(A'') \to F^{i+1}(A)$.

Contravariant δ-functors are defined accordingly. The naturality, mentioned twice, of δ^i has a precise categorical meaning: given a morphism of the exact sequence $0 \to A \to A' \to A'' \to 0$ to a sequence $0 \to B \to B' \to B'' \to 0$, the map δ^i commutes with the morphisms $F^i(A'') \to F^i(B'')$ and $F^{i+1}(A) \to F^{i+1}(B)$.

Let X be a scheme and $\Gamma(X, \bullet) : Mod(X) \to Ab$ the global sections functor. The i-th right derived functor of $\Gamma(X, \bullet)$ is by definition the cohomology functor $H^q(X, \bullet)$. If X is Noetherian then for any open affine covering $\mathcal{U}$ of X, the group $H^q(\mathcal{U}, \mathcal{F})$ of Čech cohomology with coefficients in $\mathcal{U}$ is functorially isomorphic to the sheaf cohomology $H^q(X, \mathcal{F})$; this gives an effective method to calculate cohomology in good cases.

We now will briefly obtain the *universal* properties of the right derived functors. We introduce the following definition

Definition 3.4.5. A functor F between two abelian categories is *effaceable* if for any object A there exists an object B and a monomorphism $i : A \to B$ such that $F(i) = 0$. It is *coeffaceable* if the same happens, reversing arrows, for an epimorphism.

Note that in a category with enough injectives if a functor F is such that $F(I) = 0$ then for any object A take $i : A \to I$ a morphism to an injective object I; we see that $F(i) = 0$. Thus by 3.4.3 a right derived functor of a left exact contravariant functor is effaceable.

Definition 3.4.6. A δ-functor $(F^i, \delta_i^F) : \mathcal{A} \to \mathcal{B}$, is *universal* if is such that for any other δ-functor $(T^i, \delta_i^T) : \mathcal{A} \to \mathcal{B}$ for which there is a functorial morphism $f : F^0 \to T^0$, then there are unique morphisms $f^i : F^i \to T^i$ such that $f^0 = f$ and such that for all i the diagram

$$\begin{array}{ccc} F^i(A'') & \xrightarrow{\delta_i^F} & F^{i+1}(A'') \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i+1}} \\ T^i(A'') & \xrightarrow{\delta_i^T} & T^{i+1}(A'') \end{array}$$

commutes on exact sequences $0 \to A \to A' \to A'' \to 0$.

Remark. If F^i and T^i are two universal δ-functors such that f^0 is an isomorphism between F^0 and T^0 then T^i and F^i are isomorphic. Indeed if f_i, t_i are morphisms as for definition 3.4.6 a little diagram chase shows that $\delta_F^1 = t_1 \circ f_1 \circ \delta_F^1$ and $\delta_T^1 = f_1 \circ t_1 \circ \delta_T^1$ which means $f_1 = t_1^{-1}$ and the claim follows by induction.

The next theorem gives the universality of the left derived functors

Theorem 3.4.7. *Let T^i be a covariant δ-functor, effaceable for $i > 0$. Then F^i is universal.*

Proof. [11] chap. II, 2.21. □

In particular, we have the following corollary

Corollary 3.4.8. *Let $\mathcal{A}$ be an abelian category with enough injectives, and $F : \mathcal{A} \to \mathcal{B}$ a left exact covariant functor. Then R^iF is a covariant universal δ-functor such that $R^0F \cong F$; conversely for any universal coveriant δ-functor T^i, T^0 is left exact and $T^i \cong R^iT^0$.*

Proof. Universality of R^iF follows from 3.4.7 being R^iF effaceable by proposition 3.4.3 and the following remark. For the converse note that T^0 is always left exact by the exactness of the associated long sequence, and $R^iT^0 \cong T^0$, again by 3.4.3. By the remark following universality and the first part of the corollary we have that $T^i \cong R^iT^0$. □

Of course there is an analogous of these two results for contravariant coeffaceable right derived functors, which we shall also use.

We close the section with the statement of two theorems on the vanishing of cohomology groups

Theorem 3.4.9 (Grothendieck Vanishing Theorem). *Let X be a Noetherian scheme of dimension n and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}$-modules on it. Then $H^q(X, \mathcal{F}) = 0$ for all $q > n$.*

This always reduces to a finite number the cohomology groups we have to inspect; of great importance is also

Proposition 3.4.10. *Let X be an affine scheme and $\mathcal{F}$ a coherent sheaf on X. Then we have $H^q(X, \mathcal{F}) = 0$ for $q > 0$.*

The fact that for affine schemes higher cohomology groups are trivial gives a powerful tool to calculate scheme cohomology each time it is possible to reduce to affine subsets. This theorem is the equivalent of Cartan's B Theorem in the analytic-algebraic correspondence between affine schemes and Stein manifolds.

3.5 Duality for $\mathbb{P}^n_k$

The first proposition is the basic result on cohomology of projective spaces which provides the duality in the case $\mathcal{F}$ is an invertible sheaf. Following the functors Ext^q is defined, and proofs are given of the fundamental results on these sheaves to be used in the Theorem.

Proposition 3.5.1. *Let A be a Noetherian ring and $S = A[x_0, \cdots x_n]$. Let $X = ProjS = \mathbb{P}^n_A$ be the projective n-space over A. Then* *(a)* $S \cong \bigoplus_{m\in\mathbb{Z}} H^0(X, \mathcal{O}(m))$;

(b) $H^q(X, \mathcal{O}(m)) = 0$ *for all* $0 < q < n$ *and all* $m \in \mathbb{Z}$;

(c) $H^n(X, \mathcal{O}(-n-1)) \cong A$ *and* $H^n(X, \mathcal{O}(m)) = 0$ *for all* $m > -n-1$;

(d) For all m *there is a natural non-degenerate pairing of* A*-modules*

$$H^0(X, \mathcal{O}(m)) \times H^n(X, \mathcal{O}(X, -m-n-1)) \to H^n(X, \mathcal{O}(-n-1)) \cong A.$$

Remark. As already said, since every invertible sheaf of $\mathbb{P}^n_k$ is in the form $\mathcal{O}(k)$ for a certain integer k, and all the intermediate cohomology groups are 0, this proposition establishes the complete duality under the given hypotheses, and it can be checked by taking a glance forward to the general formula (theorem 3.5.6).

Proof. We will reason in terms of Čech cohomology, which under our hypotheses is the same as sheaf cohomology. Moreover it is not restrictive to calculate homologies on a given affine covering, according to what has been explained in the previous section. The simplest choice is clearly $\mathcal{U} = \{U_i\}_{0\leq i\leq n}$ with $U_i = D_+(x_i)$. Hence, in view of the isomorphisms $H^0(\mathcal{U}, \mathcal{F}) \cong H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$, for any affine covering $\mathcal{U}$ of X and sheaf $\mathcal{F}$ on X, we just have to find the kernel of the map $C^0(\mathcal{U}, \mathcal{O}(m)) \to C^1(\mathcal{U}, \mathcal{O}(m))$.

Consider $\Gamma(U_i, \mathcal{O}(m))$: by proposition 3.3.2 this last group is just the subring of elements of degree 0 of the localization $S(m)_{x_i}$ which by definition of shifted algebra are nothing but the elements of degree m of the localization S_{x_i}, that is, fractions of type f/x_i^k, k arbitrary, and f an homogeneous polynomial of degree $m+k$. In the same way, by localizing twice (or localizing with respect to the product), for all i, j the sections $\Gamma(U_i \cap U_j, \mathcal{O}(m))$ are elements of degree m of $S_{x_i x_j}$, i.e. elements $f/x_i^k x_j^h$ with $\deg f = m + k + h$. Now we claim that the coboundary of Čech cochains

$$\delta^0 : \prod_i \Gamma(U_i, \mathcal{O}(m)) \to \prod_{i,j} \Gamma(U_i \cap U_j, \mathcal{O}(m))$$

has kernel consisting of all elements in the form $(f, \cdots, f)$, with f a polynomial of degree m, which is a subring of $\mathcal{C}^0(\mathcal{U}, \mathcal{O}(m))$ isomorphic to S_m, and this proves (a). Clearly $\delta(f, \cdots, f) = 0$; conversely the image by δ^0 of a 0-cochain is in the form $\left\{\frac{f_i x_j^{h_j} - f^j x_i^{h_i}}{x_i^{h_i} x_j^{h_j}}\right\}_{i,j}$, which is 0 if and only if $f_i = f_j x_i^{h_i}/x_j^{h_j}$ for all i, j. But f_i is a polynomial so $x_j^{h_j}$ divides f_j, which is of degree $m + h_j$; the same holds for f_i. Therefore both $f_i/x_i^{h_i}$ and $f_j/x_j^{h_j}$ are polynomials of degree m and are equal by the previous relation.

To prove (c) we must look at the last coboundary map

$$\delta^{n-1} : \prod_k \Gamma(U_{i_0\cdots\hat{i}_k\cdots i_n}, \mathcal{O}(m)) \to \Gamma(U_{i_0\cdots i_n}, \mathcal{O}(m)) \to 0$$

It is easier to consider the whole graded localization $S_{x_0\cdots x_n}$ as an algebra over A, rather than the single graded pieces $(S_{x_0\cdots x_n})_l$; this amounts to consider the sections of the sheaf $\mathcal{F} = \bigoplus_m \mathcal{O}(m)$. The coboundary above extends by additivity on $\mathcal{F}$ to a morphism

$$d^{n-1} : \hat{S} = \prod S_{x_0\cdots\hat{x}_k\cdots x_n} \to S_{x_0\cdots x_n}.$$

To obtain $H^n(X, \mathcal{F})$ we must describe the image of d^{n-1}. As a free A-module $\hat{S}$ is generated by the set $\{x_0^{h_0}\cdots\hat{x}_k\cdots x_n^{h_n}, k = 0\cdots n,\ h_i \in \mathbb{Z}\}$, whereas $S_{x_0\cdots x_n}$ is spanned by $\{x_1^{h_1}\cdots x_n^{h_n},\ h_i \in \mathbb{Z}\}$. To be a difference of rational expressions in $\hat{S}$ an element of $S_{x_0\cdots x_n}$ must have at least one $h_i \geq 0$ therefore the quotient $H^n(X, \mathcal{F})$ can be identified with those elements in the span $(x_1^{h_1}\dots x_n^{h_n})$ with all $h_i < 0$. We reconstruct the grading by observing that the m-th summand $H^n(X, \mathcal{O}(m))$ is the module generated by all the rational the elements in $H^n(X, \mathcal{F})$ such that $\sum h_i = m$. In particular for $m = -n-1$ there is only one generator, namely $1/x_0\dots x_n$, so $H^n(X, \mathcal{O}(-n-1)) \cong A$. This also shows that there are no elements in $H^n(X, \mathcal{O}(l))$ for $l > -n-1$.

The natural pairing in (d) can be now easily obtained. By (a), $H^0(X, \mathcal{O}(n))$ is just the set of homogeneous polynomials in $n+1$ variables of degree m, having basis $\{x_0^{h_1}\dots x_n^{h_n}, h_i \geq 0, \sum h_i = m\}$, while $H^n(X, \mathcal{O}(-m-n-1))$ is spanned by the rational expressions

$$\left\{x_0^{l_1}\dots x_n^{l_n}, \sum l_i = -m-n-1\right\}.$$

The pairing is then given by multiplicating monomials by monomials and letting the dual of the element $x_0^{h_1}\dots x_n^{h_n}$ be the element $x_0^{-h_1-1}\dots x_n^{-h_n-1}$.

(b) has to be proved by induction on n. It is convenient to work with the sheaf $\mathcal{F}$ rather than with the single $\mathcal{O}(m)$; this avoids each time to keep track of the degree of the elements. For $n = 1$ the statement is empty since there are no intermediate cohomologies. For $0 < q < n-1$ we have $H^q(X, \mathcal{F}) \cong 0$. We write the exact sequence

$$0 \to S(-1) \xrightarrow{x_n} S \to S/(x_n) \to 0$$

where the second map is multiplication by x_n. Now letting H be the hyperplane $D_+(x_n)$ by the exactness of the 'tilding' functor we obtain the exact sequence

$$0 \to \mathcal{O}_X(-1) \xrightarrow{\tilde{x}_n} \mathcal{O}_X \to \mathcal{O}_H \to 0$$

so that by twisting for every m in $\mathbb{Z}$ there is an exact sequence

$$0 \to \mathcal{F}(-1) \xrightarrow{\tilde{x}_n} \mathcal{F} \to \mathcal{F}_H \to 0.$$

with $\mathcal{F} \otimes \mathcal{O}_H$. For $q < n$ the long cohomology sequence associated is

$$\dots \to H^{q-1}(X, \mathcal{F}_H) \to H^q(X, \mathcal{F}(-1)) \xrightarrow{\overline{x}_n} H^q(X, \mathcal{F}) \to H^q(X, \mathcal{F}_H) \to \dots$$

For all m, by induction $H^{q-1}(X,\mathcal{F}) \cong 0$ and being $H \cong \mathbb{P}^{n-1}_A$ by Grothendieck vanishing theorem it is $H^q(X,\mathcal{F}_H) \cong H^q(H,\mathcal{F}_H(m)) \cong 0$ too, which amounts to say that $H^q(X,\mathcal{F}(-1)) \cong H^q(X,\mathcal{F})$ for all $q < n$. The isomorphism $\overline{x}_n$ is functorially obtained by the multiplication in S by x_n, which is nothing but the multiplication by the image $\overline{x}_n$ of the functorial composition used: with a notational abuse we still denote it x_n.

To conclude we restrict ourselves to the affine subscheme U_n; by proposition it is 3.4.10 $H^q(U_n,\mathcal{F}|_{U_n}) = 0$ for $q > 0$. Again by proposition 3.3.2, keeping in mind the graded structure of $\mathcal{F}$, the q-th Čech cochain of $\mathcal{F}|_{U_n}$ on the covering $\mathcal{U}' = U_1 \cap U_n, \dots, U_{n-1} \cap U_n$ is the localized cochain $C^q(\mathcal{U},\mathcal{F})_{x_n}$, whence, commuting localization and quotient,

$$H^q(X,\mathcal{F})_{x_n} = H^q(U_n,\mathcal{F}|_{U_n}) = 0.$$

This means that for all k all rationals expressions f/x_n^k with $f \in H^q(X,\mathcal{F})$ and x_n^k not dividing f, are 0 up to choosing a sufficient large k, which implies that multiplication by x_n eventually annihilates f. But this multiplication is an isomorphism and this forces $H^q(X,\mathcal{F}) = 0$ for all $0 < q < n$, which means that $H^q(X,\mathcal{O}_X(m)) = 0$ for all m and $0 < q < n$. □

We now introduce the right derived functors we are interested in

Definition 3.5.2. Let X be a Noetherian scheme, $\mathcal{F}$ a coherent sheaf on X. $\mathrm{Ext}^q(\mathcal{F},\bullet) : Coh(X) \to Ab$ is the q-th right derived functor of the left exact covariant functor $\mathrm{Hom}(\mathcal{F},\bullet)$.

When in this definition $\mathcal{F} = \mathcal{O}$ the situations quite trivializes

Proposition 3.5.3. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules over a Noetherian scheme X. Then* $\mathrm{Ext}^q(\mathcal{O},\mathcal{F}) \cong H^q(X,\mathcal{F})$ *for all q.*

Proof. We have $\Gamma(X,\bullet) \cong \mathrm{Hom}(\mathcal{O},\bullet)$. Let $\mathcal{F}$ be a coherent sheaf on X; for every open set a morphism of $\mathcal{O}(U)$-modules from $\mathcal{O}(U)$ to $\mathcal{F}(U)$ is uniquely determined by the image of one of the elements; conversely multiplication by an element of $\mathcal{F}(U)$ yields such an homomorphism. Therefore a global section $\lambda \in \Gamma(X,\mathcal{F})O$ gives raise to a sheaf morphism by considering the homomorphism $\{\lambda_U : \mathcal{O}(U) \to \mathcal{F}(U), \lambda_u(f) = f\lambda|_U\}$, and to any morphism $\{\phi_U, U \subset X\}$ from $\mathcal{O}$ to $\mathcal{F}$ remains associated the global section ϕ_X. These constructions are one the inverse of the other and functorial in nature so the isomorphism is proved. But $R^q\Gamma(X,\bullet) = H^q(X,\bullet)$ and $R^q\mathrm{Hom}(\mathcal{O},\bullet) = \mathrm{Ext}^q(\mathcal{O},\bullet)$ and the claim follows. □

Proposition 3.5.4. *Let $\mathcal{F},\mathcal{G},\mathcal{H} \in Mod(X)$ be locally free sheaves on a scheme X. Then*

$$Ext^q(\mathcal{F} \otimes \mathcal{G},\mathcal{H}) \cong Ext^q(\mathcal{F},\mathcal{G}^* \otimes \mathcal{H})$$

Proof. To prove these for $i = 0$ we use $\text{Ext}^0 \cong \text{Hom}$; this together with the isomorphisms 3.1 and 3.2 for $\mathcal{F}, \mathcal{G}, \mathcal{H}$ locally free yields

$$\text{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \text{Hom}(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})) \cong \text{Hom}(\mathcal{F}, \mathcal{G}^* \otimes \mathcal{H})$$

Now the family of functors $\text{Ext}^q(\mathcal{F} \otimes \mathcal{G}, \bullet)$ being made of derived functors is a universal δ-functor by corollary 3.4.8. Also $\text{Ext}^q(\mathcal{F}, \mathcal{G}^* \otimes \bullet)$ is a δ-functor, by composition with the exact functor $\mathcal{G}^* \otimes \bullet : Mod(X) \to Mod(X)$. Now assume that for any injective sheaf $\mathcal{I}$ and for any locally free sheaf $\mathcal{J}$ the tensor product $\mathcal{J} \otimes \mathcal{I}$ is injective; then $\text{Ext}^q(\mathcal{F}, \mathcal{G}^* \otimes \mathcal{H})$ would be 0 for any injective sheaf $\mathcal{H}$, therefore effaceable and hence universal by theorem 3.4.7. From corollary 3.4.8 we could then conclude that $\text{Ext}^q(\mathcal{F}, \mathcal{G}^* \otimes \bullet)$ is isomorphic to the right derived functors of $\text{Hom}(\mathcal{F}, \mathcal{G}^* \otimes \bullet)$ which has already been seen to be isomorphic to $\text{Hom}(\mathcal{F} \otimes \mathcal{G}, \bullet)$, and the thesis will follow.

We must then only verify $\text{Hom}(\bullet, \mathcal{J}^* \otimes \mathcal{I})$ is exact. But by the case $q = 0$ this is the same as $\text{Hom}(\bullet \otimes \mathcal{J}^*, \mathcal{I})$ which is exact because it is composition of exact functors, being $\mathcal{I}$ injective and $\mathcal{J}^*$ locally free. □

An important remark, already made, on the category $Mod(X)$ is that it *does not have enough projectives*, thus we cannot form projective resolutions of an arbitrary module $\mathcal{F}$ over X; this means that for a module $\mathcal{G}$ the functor $\text{Ext}^q(\mathcal{F}, \mathcal{G})$ in its first variable cannot be defined as the right derived functor of the contravariant left exact functor $\text{Hom}(\bullet, \mathcal{G})$. However

Lemma 3.5.5. *Let $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ be a short exact sequence of modules over X. Then we have a long exact sequence*

$$\ldots \to Hom(\mathcal{F}'', \mathcal{G}) \to Hom(\mathcal{F}', \mathcal{G}) \to Hom(\mathcal{F}, \mathcal{G}) \to Ext^1(\mathcal{F}, \mathcal{G}) \to \ldots.$$

Proof. Take an injective resolution $0 \to \mathcal{G} \to \mathcal{I}^*$. We define a functor $\mathcal{H}om(\bullet, \mathcal{I}^*) : Mod(X) \to CochAb$ from the category of modules over X to that of cochains of abelian groups by sending $\mathcal{F}$ to the long sequence

$$\ldots \to \text{Hom}(\mathcal{F}, \mathcal{I}_{n-1}) \to \text{Hom}(\mathcal{F}, \mathcal{I}_n) \to \text{Hom}(\mathcal{F}, \mathcal{I}_{n+1}) \to \ldots.$$

Moreover for fixed $\mathcal{I}_n$ the functor $\text{Hom}(\bullet, \mathcal{I}_n)$ is exact, so for an exact sequence of modules $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ the sequence of complexes

$$0 \to \text{Hom}(\mathcal{F}, \mathcal{I}_*) \to \text{Hom}(\mathcal{F}', \mathcal{I}_*) \to \text{Hom}(\mathcal{F}'', \mathcal{I}_*) \to 0$$

is exact. By taking the associated long cohomology sequence the Ext^q functors appear. □

Remark. Even if in our categories there are not enough projectives we can define Ext^q as a functor of the first variable simply by using an injective resolutions of the *second* variable, that is to say $\mathrm{Ext}^q(\bullet, \mathcal{G}) : Coh(x) \to Ab$ is the *contravariant* functor associating to a coherent sheaf $\mathcal{F}$ the group $h^q(\mathrm{Hom}(\mathcal{F}, \mathcal{I}^*))$ for a fixed injective resolution $\mathcal{I}^*$ of $\mathcal{G}$. In this fashion lemma 3.5.5 just shows that $\{\mathrm{Ext}^q(\bullet, \mathcal{G})\}_q$ is a contravariant δ-functor.

We finally prove the duality for coherent sheaves

Theorem 3.5.6 (Serre Duality for Projective Spaces). *Let k be an algebraically closed field, $X = \mathbb{P}^n_k$ and $\mathcal{K} = \mathcal{O}_X(-n-1)$. For any coherent sheaf $\mathcal{F}$ on X there is a non-degenerate pairing*

$$\phi : Hom(\mathcal{F}, \mathcal{K}) \times H^n(X, \mathcal{F}) \longrightarrow k$$

inducing an isomorphism $\theta^0 : Hom(\mathcal{F}, \mathcal{K}) \xrightarrow{\sim} H^n(X, \mathcal{K})^*$ given by

$$\theta^0(f) = \phi(f, \bullet) : H^n(X, \mathcal{F}) \to k.$$

In general for all $q > 0$ there are functorial isomorphisms

$$\theta^q : Ext^q(\mathcal{F}, \mathcal{K}) \xrightarrow{\sim} H^{n-q}(X, \mathcal{F})^*.$$

Remark. The invertible sheaf $\mathcal{K}$ is the *canonical sheaf* of the scheme X. It is defined, as in the analytic category, as the n-th external power of a certain sheaf Ω which is the scheme-theoretic equivalent of the sheaf of holomorphic 1-forms (the sections of the cotangent bundle according to our treating). Its formal definition can be given in various ways, and as one can reasonably guess, its purpose is to give a good definition of the *tangent sheaf* of a scheme. As in the analytic case, it turns out that it is also the invariant around which the Duality works, and it is a peculiar case of a *dualizing sheaf* on a scheme (see definition 3.5.7). From a purely formal viewpoint, all we need to know on $\mathcal{K}$ is that if $X = \mathbb{P}^n_k$ then $\mathcal{K} = \mathcal{O}_X(-n-1)$ (references are in [14] chap. 2.8).

Proof. Let ψ be an isomorphism between $H^n(X, \mathcal{K})$ and k as by theorem 3.5.1. By functoriality a morphism of sheaves $f \in \mathrm{Hom}(\mathcal{K}, X)$ induces a homomorphism of cohomology groups $H^n(f) : H^n(X, \mathcal{F}) \to H^n(X, \mathcal{K})$; for such an f, and $x \in H^n(X, \mathcal{F})$, ϕ is then naturally defined by $\phi(f, x) = \psi(H^n(f)(x))$. Suppose $\mathcal{F} = \mathcal{O}(k)$ for a certain $k \in \mathbb{Z}$; then by 3.5.4 and 3.5.3 (c)

$$\mathrm{Ext}^0(\mathcal{F}, \mathcal{K}) \cong \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}(-k-n-1)) \cong H^0(X, \mathcal{O}(-k-n-1))$$

and the existence of the pairing is assured by proposition 3.5.1 (d). If $\mathcal{F} = \bigoplus_{i=1}^m \mathcal{O}(k_i)$ it is, by applying again 3.5.3

$$\mathrm{Hom}(\mathcal{F}, \mathcal{K}) \cong \bigoplus_{i=1}^{m} \mathrm{Hom}(\mathcal{O}(k_i), \mathcal{K}) \cong \bigoplus_{i=1}^{m} H^0(X, \mathcal{O}(-k_i - n - 1))$$

since Hom is an addititive functor; the pairing is therefore defined summand by summand from those existing for the sheaves $\mathcal{O}(-k_i - n - 1)$. Let now $\mathcal{F}$ be an arbitrary sheaf on X;

by corollary 3.4.8 it is isomorphic to the quotient of a finite direct sum sheaf $\mathcal{E}$; adding the kernel $\mathcal{E}_0$ of the projection $\mathcal{E} \to \mathcal{F}$ we obtain the exact segment $\mathcal{E}_0 \to \mathcal{E} \to \mathcal{F} \to 0$. Now both functors $\mathrm{Hom}(\bullet, \mathcal{K})$ and $H^n(X, \bullet)^*$ are contravariant and left exact, thus we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(\mathcal{F}, \mathcal{K}) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_0, \mathcal{K}) & \longrightarrow & \mathrm{Hom}(\mathcal{E}_1, \mathcal{K}) \\ & & \downarrow{\scriptstyle \theta^0} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & H^n(X, \mathcal{F})^* & \longrightarrow & H^n(X, \mathcal{E}_0)^* & \longrightarrow & H^n(X, \mathcal{E}_1)^* \end{array}$$

with exact rows, where β and γ are isomorphisms induced by the pairing for sheaves in the forms already given. The morphism θ^0 is seen to be well defined by a little diagram chase, and the 5-lemma makes sure it is an isomorphism: therefore $Hom(\mathcal{F}, \mathcal{K}) \cong H^n(X, \mathcal{F})^*$ and $\phi(f, x) = \theta^0(f)(x)$ is the pairing we need.

Concerning the second part notice that both sides are contravariant δ-functors which by what has been just proved are isomorphic for $q = 0$. Hence by theorem 3.4.7, in its 'coeffaceable' form, and the usual reasoning concerning universality, to prove the isomorphisms all we must do is check that the two functors involved are *coeffaceable* for $q > 0$. Again, by corollary 3.4.8, the sheaf $\mathcal{F}$ can be written as a quotient of a direct sum sheaf $\mathcal{E} = \bigoplus_i \mathcal{O}(-k)$, for a certain integer $k > 0$. By 3.5.4, 3.5.3, and additivity of Ext

$$\mathrm{Ext}^q(\mathcal{E}, \mathcal{K}) \cong \bigoplus_i Ext^q(\mathcal{O}, \mathcal{K}(k)) \cong \bigoplus_i H^q(X, \mathcal{O}(k - n - 1));$$

for each $0 < q < n$ it is $H^q(X, \mathcal{O}(k-n-1)) = 0$ for all $0 < q < n$ by 3.5.1 (b); since $k > 0$ also $H^n(X, \mathcal{O}(k - n - 1)) = 0$ by 3.5.1 (c). So the δ-functor $\mathrm{Ext}^q(\bullet, \mathcal{K})$ (see the remark following lemma 3.5.5) is coeffaceable for $q > 0$, sending the surjection $\mathcal{E} \to \mathcal{F}$ to the 0 map. On the left hand it is $H^{n-q}(X, \mathcal{E})^* \cong \bigoplus_i H^{n-q}(X, \mathcal{O}(-k))^*$ by additivity of H^{n-q}, and for $0 < q < n$ the summands are 0 once more by 3.5.1 (b). Then again, by 3.5.1 (a), $\mathcal{O}(-k)$ has no global sections whence $H^{n-q}(X, \mathcal{O}(-k))^*$ vanishes for $q = n$ too; this means that also $H^{n-q}(X, \bullet)^*$ is coeffaceable for $q > 0$. □

Remark. In the case $\mathcal{F}$ is locally free it is immediate to bring the isomorphims θ^q in the more familiar form of Chapter 2. By propositions 3.5.4 and 3.5.3 (c) we have

$$\mathrm{Ext}^{n-q}(\mathcal{F}, \mathcal{K}) \cong \mathrm{Ext}^{n-q}(\mathcal{O}, \mathcal{F}^* \otimes \mathcal{K}) \cong H^{n-q}(X, \mathcal{F}^* \otimes \mathcal{K})$$

which is the equivalent of the final formula of theorem 2.2.6 with $p = 0$ when X is the projective space of dimension n on k.

We stop here our extensive investigation on Serre Duality, but before finishing we spend some more words on the further possible generalizations on general schemes. The strategy is to find a particular sheaf ω such that $H^n(X,\omega) \cong k$ which puts $\mathrm{Hom}(\mathcal{F},\omega)$ and $H^n(X,\mathcal{F})$ in duality. Such a sheaf is called a *dualizing sheaf* for X.

Definition 3.5.7. Let X be a Noetherian scheme over an algebraically closed field k. A *dualizing sheaf* for X is a pair (ω, t) with ω a coherent sheaf on X and t a *trace* morphism $t : H^n(X,\omega) \to k$ such that for every coherent sheaf $\mathcal{F}$ on X the natural pairing

$$\begin{array}{rcl} \phi : \mathrm{Hom}(\mathcal{F},\omega) \times H^n(X,\mathcal{F}) & \longrightarrow & k \\ (\alpha, x) & \longmapsto & t(H^n(\alpha), x) \end{array}$$

gives an isomorphism

$$\theta^0 : \mathrm{Hom}(\mathcal{F},\omega) \xrightarrow{\sim} H^n(X,\mathcal{F})^*.$$

It appears evident that $(\mathcal{K},\psi)$, for the isomorphism ψ as for theorem 3.5.1 (a), is a dualizing sheaf for $\mathbb{P}^n_k$. If a dualizing sheaf exists then it is unique, up to sheaf isomorphism. In general the problem of finding a dualizing sheaf on a scheme X is very difficult to resolve. On [14] for instance it is done for *projective schemes*, schemes isomorphic to a closed subscheme of $\mathbb{P}^n_k$. In that case $\omega = \mathcal{E}xt^r(\mathcal{O}_X,\mathcal{K})$, where r is the codimension of X in $\mathbb{P}^n_k$, $\mathcal{K}$ the canonical sheaf of $\mathbb{P}^n_k$, $\mathcal{O}_X$ the structure sheaf of X, and $\mathcal{E}xt^q(\mathcal{O}_X,\bullet)$ the q-th right derived functor of the left covariant functor $\mathcal{H}om(\mathcal{O}_X,\bullet) : Coh(X) \to Coh(X)$.

Grothendieck solved this problem for an even larger class of schemes, the so called *proper schemes*. Whenever we can find a dualizing sheaf, duality results analogous to theorem 3.5.6 hold; these various, more and more abstract, homological generalizations of Serre Duality led into the complete new ground of Coherent Duality Theory.

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