Target Volatility Option and claims on an asset and its realized volatility

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1. The Target Volatility Option

1.1. Motivation

- Volatility is the key ingredient of option pricing. In efficient markets, option prices reflect the market expected volatility

  1. What if most dealers are short volatility?
  2. What if risk management procedures in banks impose limits on the volatility exposure?
  3. What if there is no volatility market for the underlying? (e.g. proprietary trading strategies/indices).
  4. In periods of market stress option prices can be prohibitively high. Is there a way to take exposure to the underlying in an option without "over-paying" ?

- Target Volatility Options (TVOs) are a partial answer to the problems above. They are typically used with the following purposes:

  1. express a joint view on the performance of the asset and its volatility.
  2. cheapen the price of an option in the same way as Asian, Barrier and other Exotics.
  3. control the risk of the underlying strategy.
  4. allow dealers to buy/sell options on an underlying with no (or illiquid) volatility market.

1.2. Notation and assumptions

- For a standard Brownian motion W_t the asset price is characterized by the dynamics

$$dS_t = \sigma_t S_t dW_t. \tag{1}$$

- We shall set interest rates to zero. This is not to complicate the formulas any further, all of the following equally applies when rates are nonzero.

- The stochastic volatility process σ_t is assumed to be independent from W_t.

- Let X_t be the rescaled log-price,

$$X_t \equiv \log(S_t/S_0), \tag{2}$$

so that the realised variance $\langle X \rangle_t$ is the quadratic variation of X_t

$$\langle X \rangle_t \equiv \int_0^t \sigma_u^2 du. \tag{3}$$

- We are interested of calculating quantities of the type:

$$C_t^{TV}(K) \equiv E_t \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (S_T - K)^+ \right], \tag{4}$$

where $\bar{\sigma}$ is the Target Volatility.

1.3. TVO pricing via Taylor expansion

- Suppose one wishes to calculate the price of an ATM Call TVO at inception:

$$C_0^{TV}(S_0) \equiv E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(S_T - S_0)^+\right] \tag{5}$$

- Using the independence assumption (Hull and White, 1992) and Bachelier approximation formula, we have:

$$\begin{aligned}
C_0^{TV}(S_0) &= E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}E[(S_T - K)^+ \mid \mathcal{F}^{\sigma}{}_T]\right] && (6)\\
&= E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}C^{BS}(S_0, S_0, \langle X\rangle_T)\right] && (7)\\
&\approx S_0 E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}\sqrt{\frac{\langle X\rangle_T}{2\pi}}\right] && (8)\\
&= S_0\bar{\sigma}\sqrt{\frac{T}{2\pi}} \approx C^{BS}(S_0, S_0, \bar{\sigma}^2 T). && (9)
\end{aligned}$$

- The TVO price is approximately the Black-Scholes price of a vanilla call with implied volatility $\bar{\sigma}$.

- The idea above can be extended to a generic strike K.:

  1. Expand the Black Scholes call formula in Taylor series in K around the ATM point and look at the resulting series as a function of $\hat{\sigma} = \sigma\sqrt{t}$
  2. Divide this series by the factor $\hat{\sigma}$; the resulting series will be the deterministic version of the expansion of the payoff
  3. Substitute $\sqrt{\langle X \rangle_T}$ for $\hat{\sigma}$ take the expectation and apply linearity.
  4. By using a suitable integral representation each of the stochastic terms can be expressed in terms of the expectation of an exponential claim of the variance, which can be computed or replicated

- Expanding the Black-Scholes price around the strike K, we have

$$C(K) = C(S_0) + C^{(1)}(K - S_0) + \sum_{n=0}^{\infty} C^{(n+2)}(S_0)\frac{(K - S_0)^{n+2}}{(n+2)!} \tag{10}$$

- Derivatives higher than the second can be deduced using the formula (Estrella, 1995)

$$C^{n+2}(S_0) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\hat{\sigma}^2}{8}\right) \frac{P_n(d^+)}{S_0^{n+1}\hat{\sigma}^{n+1}}(-1)^n. \tag{11}$$

- Here $P_n(d^+)$ satisfies the following recursive equation,

$$P_n(d^+) = (d^+ + n\hat{\sigma})P_{n-1}(d^+) - P'_{n-1}(d^+), \tag{12}$$

where, if σ is the implied BS volatility

$$\hat{\sigma} = \sigma\sqrt{t}, \tag{13}$$

and

$$d^+ \equiv \frac{\log(S_0/K)}{\hat{\sigma}} + \frac{\hat{\sigma}}{2} \tag{14}$$

- Solving the equation above, rearranging things slightly and putting $\hat{\sigma}$ as a common factor, we have:

$$\begin{aligned} C(K) &= S_0 \left\{ N\left(\frac{\hat{\sigma}}{2}\right) - N\left(-\frac{\hat{\sigma}}{2}\right) \right\} \\ &- N\left(-\frac{\hat{\sigma}}{2}\right)(K - S_0) \\ &+ \exp(-\hat{\sigma}^2/8) \lim_n \left(\sum_{j=0}^{f(n)} \hat{\sigma}^{-(1+2j)} W^{n,j}(K) + O(n+3) \right), \end{aligned}$$

where

$$W^{n,j}(K) \equiv \frac{1}{\sqrt{2\pi}} \sum_{k=2j}^{n} (-1)^k C(f(k)-j,k) \frac{(K-S_0)^{k+2}}{S_0^{k+1}(k+2)!}, \tag{15}$$

$C(j,n)$ is the jth coefficient of the polynomial P_n, and

$$f(k) = \begin{cases} \frac{k}{2}, & \text{k even;} \\ \frac{k-1}{2}, & \text{k odd.} \end{cases} \tag{16}$$

- We express the volatility terms as functionals of exponential of the variance

$$\frac{1}{\sqrt{x}} N\left(-\frac{\sqrt{x}}{2}\right) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-(z+1/8)x}}{\sqrt{z+1/8}} dz, \tag{17}$$

and

$$x^{-r} = \frac{1}{r\Gamma(r)} \int_0^\infty e^{-z^{1/r}x} dz, \tag{18}$$

- Letting $\hat{\sigma} = \sqrt{\langle X \rangle_T}$ in the expansion above and substituting it in the TVO pricing formula

$$C_0^{TV}(K) = E_0\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} C^{BS}(S_0, K, \langle X \rangle_T) \right] \tag{19}$$

we obtain,

$$C_0^{TV}(K) \approx \bar{\sigma}\sqrt{T}\left[\frac{2S_0}{\sqrt{\pi}} I_0^{1/2,0} - \frac{S_0+K}{2\sqrt{\pi}}\Phi_0^{1,1/8} + \sum_{j=0}^{f(n)} \tilde{W}^{n,j}(K) I_0^{j+1,1/8}\right],$$

where the following quantities have been defined

$$\begin{aligned} I_0^{r,a} &\equiv \int_0^\infty E_0\left[e^{\lambda^{r,a}(z)\langle X\rangle_T}\right] dz \\ \Phi_0^{r,a} &\equiv \int_0^\infty \frac{E_0\left[e^{\lambda^{1,1/8}(z)\langle X\rangle_T}\right]}{\sqrt{z+1/8}} dz, \end{aligned}$$

$$\lambda^{r,a}(z) \equiv -(z^{1/r}+a), \quad \tilde{W}^{n,j} \equiv \frac{W^{n,j}(K)}{(j+1)!}. \tag{20}$$

- Alternatively, for short dated maturities, we can use the Bachelier approximation:

$$C_0^{TV}(K) \approx S_0\bar{\sigma}\sqrt{\frac{T}{\pi}} + \bar{\sigma}\sqrt{T}\left[\frac{K-S_0}{2\sqrt{\pi}}\Phi_0^{1,1/8} + \sum_{j=0}^{f(n)} \tilde{W}^{n,j}(K) I_0^{j+1,1/8}\right]. \tag{21}$$

- The expressions above can be calculated in closed-form for a variety of models (e.g. affine stochastic volatility models).

1.4. TVO pricing via Taylor expansion, $t > 0$

- The pricing problem at $t > 0$ is similar, but some symmetry is lost:

$$C_t(K) \equiv E_t\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(S_T - K)^+\right] \tag{22}$$

$$= E_t\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\epsilon_t + \langle X\rangle_T - \langle X\rangle_t}} C^{BS}(S_t, K, \langle X\rangle_T - \langle X\rangle_t)\right], \tag{23}$$

where $\epsilon_t \equiv \langle X\rangle_t$.

- The Black-Scholes Call price can again be expanded via Taylor around S_t, but this time we have to deal with objects of the following form:

$$q_1(x) \equiv \frac{N(-\sqrt{x}/2)}{\sqrt{\epsilon + x}}, \tag{24}$$

$$q_2(x) \equiv \frac{x^{-(j+1/2)}}{\sqrt{\epsilon + x}}. \tag{25}$$

- To price this resulting decomposition claims we have to further expand the functions (24) and (25) a second time in ϵ. This will yield a double convergent series: by considering a finite truncation, the summation can be rearranged as a linear combination of claims on volatility

- The price of the TVO at time t is therefore of the form

$$C_t^{TV}(K) \approx \bar{\sigma}\sqrt{T}\left[\frac{2S_t}{\sqrt{\pi}} I_t^{1/2,0,0} - \frac{S_t + K}{2\sqrt{\pi}} \Phi_t^{1,1/8} + \sum_{j=0}^{m+f(n)} \hat{W}^{n,m,j}(K, \langle X \rangle_t) I_t^{j+1,1/8,0}\right]. \tag{26}$$

- Similarly to the results of the previous section, the TVO price is a linear combination with weights $W^{n,m,j}(K, \langle X \rangle_t)$ of terms of the form

$$\begin{aligned}
I_t^{r,a,b} &= \int_0^\infty e^{-(z^{1/r}+b)\epsilon_t} E_t\left[e^{\lambda^{r,a}(\langle X \rangle_T - \langle X \rangle_t)}\right] dz, \\
\Phi_t^{1,a} &= \int_0^\infty \frac{e^{-(z+a)\epsilon_t}}{\sqrt{z+a}} E_t\left[e^{\lambda^{1,a}(\langle X \rangle_T - \langle X \rangle_t)}\right] dz,
\end{aligned}$$

for some non-negative real constants a and b, and

$$\lambda^{r,a} \equiv -(z^{1/r} + a). \tag{27}$$

1.5. Robust pricing

- One of the remarkable features of the Taylor expansion technique is that allows completely model-independent pricing via replication of the volatility claims $I_t^{\cdot}$ and $\Phi_t^{\cdot}$ in the finite sum (26)
- Carr-Lee provide a way to express the exponential of the quadratic variation in terms of the terminal value of the underlying. For any complex λ we have:

$$E_t[e^{\lambda(\langle X\rangle_T - \langle X\rangle_t)}] = E_t\left[e^{(X_T - X_t)p(\lambda)}\right] = E_t\left(\frac{S_T}{S_t}\right)^{p(\lambda)}, \tag{28}$$

where

$$p(\lambda) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2\lambda}. \tag{29}$$

- Breeden-Litzenberger (1978) show that sufficiently smooth payoffs f can be expressed as a portfolio of calls and puts:

$$f(S) = f(k) + f^{'}(k)[S-k] + \int_k^{\infty} f^{''}(x)(S-x)^+ dx + \int_0^k f^{''}(x)(x-S)^+ dx \quad (30)$$

- The Carr-Lee formula can be used to express $I_t^{\cdot}$ and $\Phi_t^{\cdot}$ in the Taylor expansion for $t > 0$ as a function of S_T:

$$\begin{aligned} I_t^{r,a,b} &= \int_0^{\infty} e^{-(z^{1/r}+b)\langle X\rangle_t} E_t\left[e^{\lambda^{r,a}(\langle X\rangle_T - \langle X\rangle_t)}\right] = E_t \int_0^{\infty} e^{-(z^{1/r}+b)\langle X\rangle_t} \text{Re}\left(\frac{S_T}{S_t}\right)^{p^{r,a}(z)} dz \\ \Phi_t^{1,a} &= \int_0^{\infty} \frac{e^{-(z+a)\langle X\rangle_t}}{\sqrt{z+a}} E_t\left[e^{\lambda^{1,a}(\langle X\rangle_T - \langle X\rangle_t)}\right] dz = E_t \int_0^{\infty} \frac{e^{-(z+a)\langle X\rangle_t}}{\sqrt{z+a}} \text{Re}\left(\frac{S_T}{S_t}\right)^{p^{1,a}(z)} dz, \end{aligned}$$

where

$$p^{r,a} \equiv 1/2 \pm \sqrt{1/4 - 2z^{1/r} - 2a}. \quad (31)$$

- In particular, we can define the functions

$$\begin{aligned}\tilde{I}^{r,a,b}(S) &\equiv \int_0^\infty e^{-(z^{1/r}+b)\epsilon_t}\mathrm{Re}\left(\frac{S}{S_t}\right)^{p^{r,a}(z)} dz\\ \tilde{\Phi}(S)^{1,a} &\equiv \int_0^\infty \frac{e^{-(z+a)\epsilon_t}}{\sqrt{z+a}}\mathrm{Re}\left(\frac{S}{S_t}\right)^{p^{1,a}(z)} dz.\end{aligned}$$

- $I^{r,a,b}(S)$ and $\Phi(S)^{1,a}(S)$ are twice differentiable in S and well defined in S_t. We can thus use the Breeden-Litzenberger representation (30) to express the price of a TVO as a function of traded instruments.

- It is quite a major inconvenience that at inception robust pricing cannot be carried out. Indeed when $t = 0$ the claims $I_0^{r,a}$ and $\Phi_0^{r,a}$ cannot be written as an expectation of an integral.

1.6. TVO pricing using Laplace Transforms, the $t > 0$ case

- Let's consider the pricing problem of a put TVO, where the pay-off is expressed in terms of the log-strike k and the log-terminal value s_T

$$P_t(k) \equiv E_t\left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X\rangle_T}}(e^k - e^{s_T})^+\right] \tag{32}$$

- For any complex α such that $\mathrm{Re}(\alpha) > 1$, the Laplace transform of $P(k)$ is equal to

$$\begin{aligned}\hat{P}_t(\alpha) &\equiv \int_0^\infty e^{-\alpha k} P_t(k) dk \\ &= \bar{\sigma}\sqrt{T} S_t^{1-\alpha} E_t \left[\frac{1}{\sqrt{\epsilon_t + \langle X \rangle_T - \langle X \rangle_t}} \frac{e^{(1-\alpha)(X_T - X_t)}}{\alpha(\alpha-1)} \right].\end{aligned}$$

- The denominator admits the familiar representation

$$\frac{1}{\sqrt{\epsilon + x}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2(\epsilon + x)} dz. \tag{33}$$

- Using the independence of σ and W_t together with Fubini's Theorem, we can write $\hat{P}(\alpha)$ in terms of S_t and the quadratic variation

$$\hat{P}_t(\alpha) = 2\bar{\sigma}\sqrt{\frac{T}{\pi}} S_t^{1-\alpha} \int_0^\infty \frac{e^{-z^2\langle X \rangle_t} E_t\left[e^{\lambda_{z,\alpha}(\langle X \rangle_T - \langle X \rangle_t)}\right]}{\alpha(\alpha-1)} dz \tag{34}$$

  where $\lambda_{z,\alpha} = -(z^2 + \alpha(1-\alpha))$

- If we don't mind model dependence, we can calculate explicitly the quantities inside the expectation for a variety stochastic volatility models (e.g., affine models).

- The price of the TVO options can be obtained by numerically inverting the closed form Laplace transform (34).

1.7. Numerical Results

- In the following numerical examples we assumed the following Heston model:

$$\begin{aligned} dS_t &= v_t^{1/2} S_t dW_t, \\ dv_t &= \kappa(\theta - v_t)dt + \eta v_t^{1/2} dZ_t, \end{aligned} \tag{35}$$

and parameters

$$S_0 = 100, \quad v_0 = 0.2, \quad \overline{\sigma} = 0.1, \quad \kappa = 0.5, \quad \theta = 0.2, \quad \eta = 0.3. \tag{36}$$

Table 1: **Maturity $T = 3$, $t = 0$. TVOs prices for different strikes, using different pricing methods**

Strike	Taylor polynomial of order n					Laplace Transform	Monte Carlo simulation
	n=1	n=2	n=3	n=4	n=5		
60	10.1534	11.1768	11.3814	11.4006	11.3958	11.3919	11.3897
80	8.4475	8.7033	8.7289	8.7301	8.7300	8.7301	8.7281
100	6.7416	6.7416	6.7416	6.7416	6.7416	6.7416	6.7415
120	5.0357	5.2915	5.2659	5.2671	5.2673	5.2672	5.2618
140	3.3298	4.3532	4.1485	4.1677	4.1725	4.1699	4.1643

Table 2: **Maturity $T = 5$, $t = 2$, $K = 100$, $S_t = 120$. TVO prices for various realized volatility levels.**

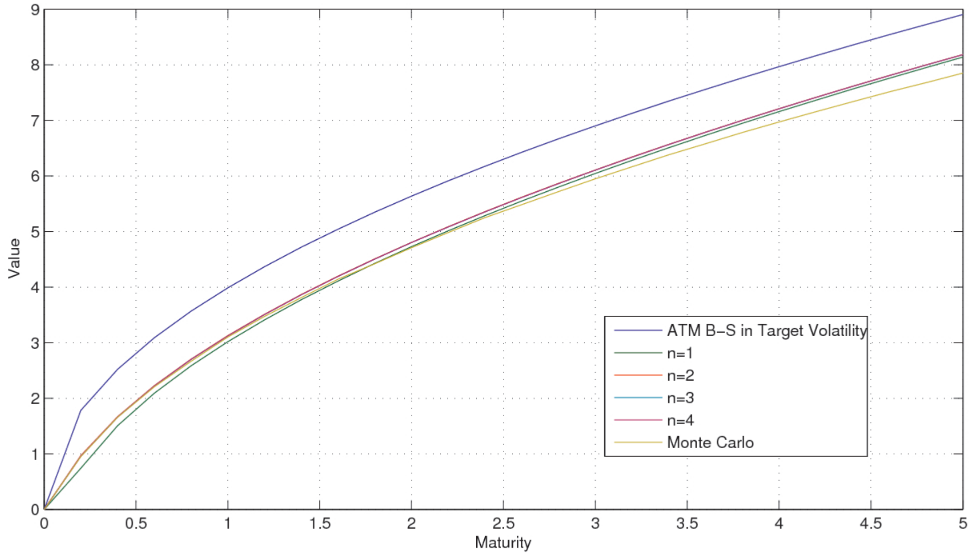
$\langle X \rangle_t$	Taylor polynomial of order n					Laplace Transform	Monte Carlo simulation
	n=1	n=2	n=3	n=4	n=5		
0.2	10.7404	10.9632	10.9817	10.9828	10.9828	10.9780	10.9772
0.4	9.5530	9.7438	9.7597	9.7607	9.7607	9.7416	9.7391
0.6	8.7105	8.8795	8.8935	8.8945	8.8945	8.8601	8.8577
0.8	8.0694	8.2222	8.2349	8.2358	8.2358	8.1872	8.1894
1	7.5591	7.6993	7.7110	7.7118	7.7118	7.6503	7.6416

Table 3: **Maturity $T = 4$, $t = 3$, $S_0 = 100$, $\langle X \rangle_t = 0.3$. TVOs prices for different strikes**

Strike	Taylor polynomial of order n					Laplace Transform	Monte Carlo simulation
	n=1	n=2	n=3	n=4	n=5		
60	9.7467	11.4962	11.8461	11.8810	11.8740	11.9816	11.9827
80	7.4080	7.8454	7.8892	7.8913	7.8911	7.8390	7.8400
100	5.0694	5.0694	5.0694	5.0694	5.0694	4.9175	4.9067
120	2.7307	3.1681	3.1243	3.1265	3.1267	3.0313	3.0338
140	0.3920	2.1415	1.7916	1.8266	1.8336	1.8690	1.8680

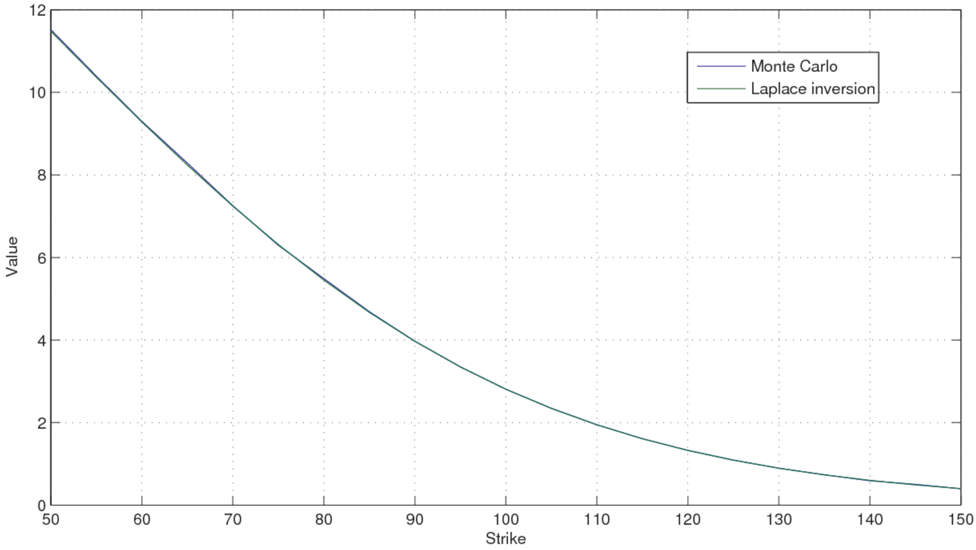
1.7.1. Taylor expansion

Figure 1: Value of the TVO as a function of the time to maturity. K=110.



1.7.2. Laplace transform

Figure 2: Laplace transform Call TVO prices as a function of the strike, T=0.5.



2. Claims on an asset and its realised variance

- Volatility derivatives have been extensively studied and a great deal is known about pricing and hedging them. Recently, products on volatility adjusted indices took off as a way of controlling exposure to volatility of the underlying.
- Motivated by this we are concerned in studying a class of such products written in the form $F(S_T, \langle X \rangle_T)$. The TVO clearly falls under this class.

2.1. Pricing via the associated PDE

- Assume that in a risk neutral equivalent measure the dynamics for the asset and the stochastic volatility are given by

$$\begin{aligned} dS_t &= rS_t + \sqrt{v_t} S_t dW_t^1 \\ dv_t &= \alpha_t dt + \beta_t dW_t^2. \end{aligned} \tag{37}$$

Here r is a riskless constant rate, and W^1 and W^2 are Brownian motions of constant correlation ρ. The quadratic variation evolves according

$$dI_t = v_t dt$$

- It can be shown, by the usual no-arbitrage/hedging argument, that given a payoff $F_T = F(S_T, \langle X \rangle_T)$ its time t value

$$V(S_t, I_t, v_t, t) = E_t[e^{-r(T-t)} F_T]$$

is the only solution of the partial differential equation

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \alpha_t \frac{\partial V}{\partial v} + v_t \frac{\partial V}{\partial I} + \frac{v_t S_t^2}{2} \frac{\partial^2 V}{\partial^2 S} + \frac{\beta_t^2}{2} \frac{\partial^2 V}{\partial^2 v} + \rho \beta_t \sqrt{v_t} S_t \frac{\partial V}{\partial S \partial v} - rV = 0 \quad (38)$$

if a solution to such equation does exist.

- For some relevant stochastic volatility models this equation admits a fundamental transform. A fundamental transform is a solution for the Fourier-transformed equation with respect to the variables $x = \log S_t + r(T-t)$ and I_t, dampened by $r(T-t)$ and with terminal condition 1. Indeed setting

$$\hat{W}(\omega, \eta, v, \tau) = \int_{\mathbb{R}^2} e^{ix\omega + iI\eta} e^{-r(T-t)} V(e^x, I, v, T-t) dx dI,$$

we see that $\hat{W}$ satisifes

$$\frac{\beta^2}{2} \frac{\partial^2 \hat{W}}{\partial^2 v} + \frac{\partial \hat{W}}{\partial v} (\alpha - i\omega \sqrt{v} \rho \beta) - \frac{v}{2} (\omega^2 - i\omega + 2i\eta) \hat{W} = \frac{\partial \hat{W}}{\partial \tau} \quad (39)$$

Such an equation with $\hat{W}(\omega, \eta, v, 0) = 1$ can be solved as in Lewis (2000).

- Call $\hat{H}$ the fundamental transform and let

$$\hat{W}(\omega, \eta, v, 0) = \int_{\mathbb{R}^2} e^{ix\omega + iI\eta} V(e^x, I, v, 0) dx dI$$

be the accordingly transformed payoff. The solution for the general problem is then given by the inverse Fourier transform of H times $\hat{W}(\omega, \eta, v, 0)$

$$V(S, I, v, t) = \frac{e^{-r(T-t)}}{4\pi^2} \tag{40}$$

$$\int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} S^{-i\omega} e^{-i\omega(r-d)(T-t)} e^{-i\eta I} \hat{H}(\omega, \eta, v, T-t) \hat{W}(\omega, \eta, v, 0) d\omega d\eta. \tag{41}$$

- One remarkable feature of this technique is that allows to effectively separate, by means $\hat{H}$ and $\hat{W}(\tau = 0)$, the price component due to the payoff and that about the stochastic volatility model.

- Although this is a model-dependent solution it works with any model and allows correlation.

- This representation also allows the computation of hedging ratios. It will be sufficient to differentiate in S under integral sign.

- Numerical testing confirms the correctness of this method. Let us revise Table 1 and compare it with the TVO prices under this new method:

Table 4: **Maturity $T = 3$, $t = 0$. TVOs prices for different strikes**

Strike	Taylor polynomial of order n			Laplace Transform	Monte Carlo simulation	PDE pricing
	n=3	n=4	n=5			
60	11.3814	11.4006	11.3958	11.3919	11.3897	11.3909
80	8.7289	8.7301	8.7300	8.7301	8.7281	8.7299
100	6.7416	6.7416	6.7416	6.7416	6.7415	6.7415
120	5.2659	5.2671	5.2673	5.2672	5.2618	5.2672
140	4.1485	4.1677	4.1725	4.1699	4.1643	4.1699

- However more general settings are now possible. With the usual parameters:

Table 5: $S_t = 100$, $K = 85$, $\langle X \rangle_t = 0.46$, $T = 5$, $t = 2.5$, $r = 0.08$. **TVO price via PDE for various correlation levels**

Correlation	Monte Carlo	PDE pricing
-0.8	10.3154	10.3975
-0.4	9.9415	9.9505
0	9.4398	9.4549
0.4	8.9645	8.9059
0.8	8.3136	8.3025

2.2. Replication of $F(S_T, \langle X \rangle_T)$ by a volatility derivative

- We now revert to the assumption of 0 correlation between the Brownian motion driving S_t and the stochastic volatility σ_t.

- In such a situation it is possible to carry out the reduction operated for the TVO for general payoffs. Again we use that under independence the time t conditional distribution of S_T/S_t is log-normal with expectation

$$r(T-t) - (\langle X \rangle_T - \langle X \rangle_t)/2$$

  and variance

$$\langle X \rangle_T - \langle X \rangle_t.$$

- Exactly as in (42) by conditioning with respect to the whole path followed by σ_t, and then using the fact that the inner conditional distribution is lognormal,

$$\begin{aligned} E_t\left[e^{-r(T-t)} F(S_T, \langle X \rangle_T)\right] &= E_t\left[e^{-r(T-t)} E_t[F(S_T, \langle X \rangle_T) \mid \mathcal{F}^{\sigma}{}_T]\right] \\ &= E_t\left[e^{-r(T-t)} F^{BS}(S_t, \langle X \rangle_T)\right]. \end{aligned}$$

- $F^{BS}(S_t, \langle X \rangle_T)$ stands for the Black-Scholes price in the first argument, of spot S_t and variance $\langle X \rangle_T$. We call this claim the "variance equivalent replicating payoff".

- We then see that in absence of correlation pricing F amounts to the valuation of a claim on the variance only. In turn such a claim can priced by one of the many methods available in the literature (Carr and Lee).

- If correlation is allowed then the previous formula does not holds exactly; nevertheless it represents a first order approximation for the price:

$$E_t\left[e^{-r(T-t)}F(S_T,\langle X\rangle_T)\right]=E_t\left[e^{-r(T-t)}F^{BS}(S_t,\langle X\rangle_T)\right]+O(\rho).$$

3. More derivatives

- Interesting payoffs can now be considered, besides the TVO. They are not actually traded but they all, at least mathematically, make sense.
- **Target Volatility Option**
  - Payoff
$$F(S_T, \langle X \rangle_T) = \frac{\overline{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}}(S_T - K)^+$$
  - Fourier transform in log-strike and variance
$$\hat{F}(\omega, \eta) = \frac{\overline{\sigma}(1+i)\sqrt{\pi T}}{\sqrt{2\eta}} \frac{K^{1+i\omega}}{(i\omega - \omega^2)} \quad \text{if } Im(\omega) > 1\,, Im(\eta) > 0$$
  - Variance equivalent replicating payoff
$$\frac{\overline{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} Call^{BS}(S_0, K, \langle X \rangle_T)$$

- **"Digital Asset-Variance Call"**
  - Payoff
$$F(S_T, \langle X\rangle_T) = \chi_{\{S_T \geq K_1, \langle X\rangle_T \geq K_2\}}$$
  - Fourier transform in log-strike and variance
$$\hat{F}(\omega, \eta) = -\frac{K_1^{i\omega} e^{iK_2\eta}}{\omega\eta} \quad \text{if } Im(\omega) > 0\,, Im(\eta) > 0$$
  - Variance equivalent replicating payoff
$$DigCall^{BS}(S_0, K_1, \langle X\rangle_T)\chi_{\{\langle X\rangle_T \geq K_2\}}$$
- **"Asset-Variance exchange option"**
  - Payoff
$$F(S_T, \langle X\rangle_T) = (S_T - N\langle X\rangle_T)^+$$
  - Fourier transform in log-strike and variance
$$\hat{F}(\omega, \eta) = N(-i\eta)^{-2-i\omega}\frac{\Gamma(2+i\omega)}{i\omega - \omega^2} \quad \text{for } Im(\omega) > 1,\ Im(\eta) < 2$$
  - Variance equivalent replicating payoff
$$Call^{BS}(S_0, \langle X\rangle_T, \langle X\rangle_T)$$

- **"Target Volatility Barrier Option"**

  - Payoff

$$F(S_T, \langle X \rangle_T) = \frac{\overline{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}}(S_T - K)^+ \chi_{\{Max(S_t)<\kappa_1, \langle X \rangle_T < \kappa_2\}}$$

  - Fourier transform in log-strike and variance

$$\hat{F}(\omega, \eta) = \frac{\overline{\sigma}(1+i)\sqrt{\pi T}}{\sqrt{2\eta}} \frac{K^{1+i\omega}}{(i\omega - \omega^2)} \quad \text{if } Im(\omega) > 1\,, Im(\eta) > 0$$

- These derivatives have been partially tested: the PDE and the variance claim replication prices match with the Monte Carlo simulation in all cases studied

4. Further Work

- Hedging.
- Non zero correlation case.
- Target volatility indices:

$$\frac{\Delta I_t}{I_t} = \frac{\bar{\sigma}\sqrt{\Delta T}}{\sqrt{\langle X\rangle_{t+\Delta t} - \langle X\rangle_t}} \frac{\Delta S_t}{S_t} \tag{42}$$

$$\approx \bar{\sigma}\Delta W_t \tag{43}$$

- The price of call option on a TVO index is thus approximately

$$E_0[(I_T - K)^+] = C^{BS}(I_0, K, \bar{\sigma}) + \mathrm{Err}(volvol, \rho, \Delta t) \tag{44}$$