Volatility targeting using delayed diffusions

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Target Volatility

A target volatility strategy (TVS) is a bond-equity portfolio strategy aimed at keeping the instantaneous volatility of the investment constant at a target $\overline{\sigma}$. The belief is that this can be achieved by allocating the equity exposure as an inverse proportion of the equity volatility. As volatility rises the portfolio is shielded in the bond to avoid losses. When it drops back, the equity exposure is increased to benefit of the upside.

Motivation

In a market with stochastic volatility, consider a diffusion model for the asset S_t and bond B_t:

$$dB_t = rB_t dt, \quad dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dW_t.$$

The fund $d\pi_t = \Theta_t dS_t + \psi_t dB_t$ in terms of the equity weight $w_t = \Theta_t S_t / \pi_t$ has dynamics

$$\begin{aligned} d\pi_t &= \Theta_t dS_t + \psi_t dB_t = w_t \pi_t dS_t / S_t + (1 - w_t)\pi_t dB_t / B_t \\ &= r\pi_t dt + \pi_t \, \overline{\sigma}(\mu_t - r)/\sqrt{v_t} dt + \overline{\sigma}\pi_t dW_t \end{aligned}$$

after choosing $w_t = \overline{\sigma}/\sqrt{v_t}$. Then π_t has constant variance, and derivatives on π_t have Black-Scholes values.

However v_t is not observable, and asset managers use instead the realized variance with an estimator $h(t)$ over a time window δ, possibly capped at a maximum exposure C:

$$w_t^{h,\delta,C} = \min\left\{ C, \overline{\sigma}\left(\int_0^\delta h(u) v_{t-u} du \right)^{-1/2} \right\}.$$

We then have a family of *stochastic delayed differential equations* (SDDEs) $\pi_t^{h,\delta,C}$ in place of the simple SDE for π_t above.

In this realistic setup, the correctness of Black-Scholes valuations on a TVS must be demonstrated

Main Results

Proposition 1. *If $\mathbb{Q}$ is a pricing measure under which $e^{-rt}S_t$ is a martingale, then $e^{-rt}\pi_t^{h,\delta,C}$ is also a martingale under $\mathbb{Q}$.*

Theorem 1. *Under $\mathbb{Q}$, assume that there exist $p > 1$ and $q > p/(p-1)$ such that:*

1. *v_t is in $L^p(\Omega \times [0,T])$;*
2. *the sequence $(w_t^{h,\delta,C})^2$ is bounded in $L^q(\Omega \times [0,T])$.*

Let

$$X_t = \log(\pi_0) + (r - \overline{\sigma}^2/2)t + \overline{\sigma}W_t :$$

we have that $\lim_{\delta \to 0, C \to \infty} \log(\pi_t^{h,\delta,C}) = X_t$ *in* $L^1(\Omega)$, $\forall t \leq T$

Combining these two results we conclude that for reasonable δ and C, Black-Scholes valuation is possible for forward contracts and call and put options. Also, the statement of the theorem is non-empty because of the following:

Proposition 2. *The stochastic variance model by Heston and the 3/2 model with $h(t), \delta$ being a sequence of equally weighting or EWMA volatility estimators, satisfy the assumptions of the theorem.*

Numerical Methodology

We have numerically tested our theoretical findings with a Monte Carlo simulation using an *ad hoc* numerical scheme based on the SDDEs Markovian approximation of [2]. This allowed faster and more accurate numerical simulations. The equity S_t follows the Heston model.

Path Simulations

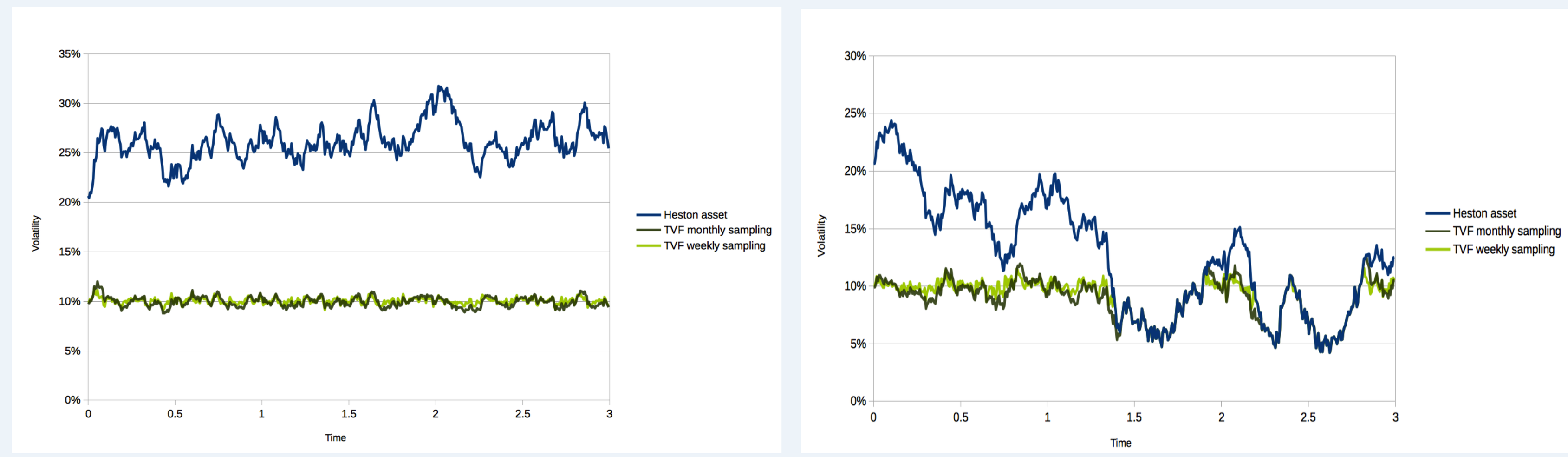

Figure 1: On the left the volatility realizes at high level, and the TVS works as expected. On the right, the volatility plunges and the cap C=1 is hit.

Volatility Surfaces

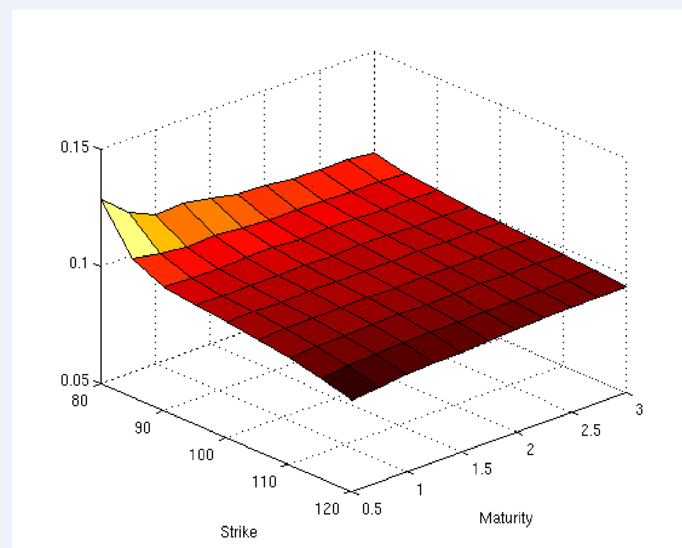

Figure 2: Volatility surfaces of options on TVS with equal weighting estimator h and weekly estimation window δ, roughly constant at $\overline{\sigma} = 0.1$.

Vega

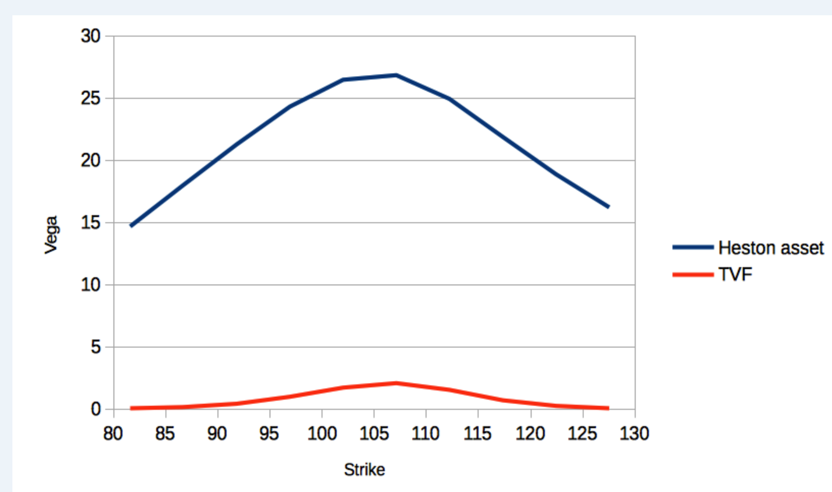

Figure 3: Vega of S_t compared to that of a TVS. Consistently with its Black-Scholes character, the vega of the TVS is nearly 0.

Conclusion

In some popular stochastic volatility models the value of contingent claims on TVSs is nearly Black-Scholes. Therefore, in these models, it is possible to easily and transparently assess return guarantees and hedge costs of target volatility funds.

References

[1] L. Torricelli, "Target volatility investment strategies using stochastic delayed differential models," *Submitted to SIFIN*, 2017.

[2] S. Federico and P. Tankov, "Finite-dimensional representations for controlled diffusions with delay," *Applied Mathematics and Optimization*, vol. 71, pp. 165–194, 2015.