Financial products depending jointly on an asset and its volatility: case studies and a theoretical view

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June 20, 2013

Workplan

Introduction

In recent times investors have shown interest in financial derivatives/structured products built on the joint interplay of an underlying asset (equity) S_t and the realized volatility RV_t this same asset accrues over a time period.

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- To simplify the valuation framework in sophisticated market environments;
- To express joint views of equity and volatility performances;
- To be able to in trade vanilla type instruments when markets are under stress, in particular when high implied volatilities give rise to unaccessible prices.

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1. The target volatility option, a joint asset and volatility derivative written in an option format;
2. The target volatility strategy; a structured investment strategy consisting in rebalancing the equity proportion in a portfolio with the volatility it realizes;
3. The valuation problem of a general joint asset/volatility derivative in a stochastic volatility model, and produce risk-neutral pricing formulae.

The target volatility option

In a market setting where no interest rate is paid on bonds, let S_t be an equity modeled through a stochastic volatility process σ_t, i.e.:

$$dS_t = \sigma_t S_t dW_t \tag{1}$$

for a Wiener process W_t. The realized variance of S_t is the annualized quadratic variation of $X_t = \log S_t$, i.e. the quantity

$$\frac{\langle X \rangle_t}{t} = \frac{1}{t} \int_0^t \sigma_s^2 ds \tag{2}$$

This is the usual continuous-time proxy of the annualized variance of the equity log-returns compounded in $[0, t]$. We assume also that the processes S_t and σ_t are independent.

The realized volatility RV_t is thus simply $\sqrt{\langle X \rangle_t / t}$.

The target volatility option

The target volatility option (TVO) is a European-type derivative contract written on S_t and $\langle X \rangle_t$ paying off at maturity T:

$$TVO_T = \frac{\overline{\sigma}}{RV_T}(S_T - K)^+ = \overline{\sigma}\sqrt{\frac{T}{\langle X \rangle_T}}(S_T - K)^+ \tag{3}$$

for a fixed strike price K and a pre-specified target volatility level $\overline{\sigma}$ written in the contract.

We shall show two pricing results for the TVO valuation problem:

1. An approximated formula based on a Taylor series expansion of the Black-Scholes call payoff in the variable K around S_0;

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1. An approximated formula based on a Taylor series expansion of the Black-Scholes call payoff in the variable K around S_0;
2. A semi-closed integral formula based on the inversion of the Laplace transform of the payoff in the log-strike.

The target volatility option

The motivation for such a contract is as follows. Assume that an at-the-money ($K = S_0$) TVO is traded. By taking risk-neutral expectations and using the conditioning trick and the Bachelier approximation for the Call price yields:

$$\mathbb{E}\left[\overline{\sigma}\sqrt{\frac{T}{\langle X\rangle_t}}(S_T - S_0)^+\right] = \mathbb{E}\left[\left[\overline{\sigma}\sqrt{\frac{T}{\langle X\rangle_t}}(S_T - S_0)^+|\sigma_T\right]\right] \tag{4}$$

$$= \mathbb{E}\left[\sqrt{\frac{T}{\langle X\rangle_t}}C^{BS}(S_0, S_0, \langle X\rangle_t)\right] \sim \mathbb{E}\left[\sqrt{\frac{T}{\langle X\rangle_t}}\frac{\langle X\rangle_t}{\sqrt{2\pi}}\right] = \overline{\sigma}\sqrt{\frac{T}{2\pi}} = \tag{5}$$

$$C^{BS}(S_0, S_0, \overline{\sigma}) \tag{6}$$

that is, the price of an at-the-money TVO having target volatility $\overline{\sigma}$ is approximately the at-the-money Black-Scholes call price of implied volatility $\overline{\sigma}$.

What if $K \neq S_0$ ranges in a neighbourhood of S_0?

The target volatility option: Taylor series expansion

If we take up equation (4) and let $K \neq S_0$, we see again that due to independence of S_t and $\langle X \rangle_t$ pricing a TVO is equivalent to pricing the a volatility derivative $\sqrt{\frac{T}{\langle X \rangle_t}} C^{BS}(S_0, K, \langle X \rangle_t)$, i.e.:

$$TVO_0 = \mathbb{E}\left[\overline{\sigma}\sqrt{\frac{T}{\langle X \rangle_T}} C^{BS}(S_0, K, \langle X \rangle_T)\right] \tag{7}$$

To obtain a price we proceed as follows:

1. We develop $C^{BS}(S_0, K, x)$ as a function of K in its Taylor series around S_0 and multiply the series thus obtained by the factor $\overline{\sigma}\sqrt{\frac{T}{x}}$;

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3. We truncate such series, compute it at $x = \langle X \rangle_T$, take the risk neutral expectation, apply Fubini's theorem;

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3. We truncate such series, compute it at $x = \langle X \rangle_T$, take the risk neutral expectation, apply Fubini's theorem;
4. What we are left with is a linear combination of integrals of the Laplace transform of the the volatility model proxying the price.

The target volatility option: Taylor series expansion

Lemma

The Black and Scholes (call) equation admits the following Taylor expansion as a function of the strike K around the ATM point S:

$$C^{BS}(S,K,x) = S - (S+K)N\left(-\frac{\sqrt{x}}{2}\right) + e^{-x/8}\sum_{j=0}^{f(n)} x^{-(1/2+j)} W^{n,j}(K) + O(n+3), \tag{8}$$

where

$$W^{n,j}(K) \equiv \frac{1}{\sqrt{2\pi}} \sum_{k=2j}^{n} (-1)^k c^{f(k)-j,k} \frac{(K-S)^{k+2}}{S^{k+1}(k+2)!}, \tag{9}$$

and

$$f(k) = \begin{cases} \frac{k}{2}, & k \text{ even;} \\ \frac{k-1}{2}, & k \text{ odd.} \end{cases} \tag{10}$$

Now we multiplicate equation (8) by $\overline{\sigma} T / \sqrt{x}$ and make use of the following integral representations:

$$\frac{1}{\sqrt{x}} N\left(-\frac{\sqrt{x}}{2}\right) = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-(z+1/8)x}}{\sqrt{z+1/8}} dz, \tag{11}$$

and, for $r > 0$:

$$x^{-r} = \frac{1}{r\Gamma(r)} \int_0^\infty e^{-z^{1/r}x} dz. \tag{12}$$

By substituting equation (8) into (7), truncating the series, using (11) and (12), and finally passing the expectation inside the integral, we have the following Proposition:

The target volatility option: Taylor series expansion

Proposition

The price of a call TVO can be approximated by a linear combination of integrals of some exponential function of the quadratic variation,

$$C_0^{TV}(K) \approx \bar{\sigma}\sqrt{T}\left[\frac{2S_0}{\sqrt{\pi}} I_0^{1/2,0} - \frac{S_0 + K}{2\sqrt{\pi}} \Phi_0^{1,1/8} + \sum_{j=0}^{f(n)} \frac{W^{n,j}(K)}{(j+1)!} I_0^{j+1,1/8}\right], \tag{13}$$

$$I_0^{r,a} = \int_0^{\infty} \mathbb{E}\left[e^{\lambda^{r,a}(z)\langle X\rangle_T}\right] dz \tag{14}$$

$$\Phi_0^{r,a} = \int_0^{\infty} \frac{\mathbb{E}\left[e^{\lambda^{1,1/8}(z)\langle X\rangle_T}\right]}{\sqrt{z+1/8}} dz, \tag{15}$$

$$\lambda^{r,a}(z) = -(z^{1/r} + a). \tag{16}$$

The target volatility option: Taylor series expansion, $t > 0$

If we want to evaluate the TVO after inception $\overline{\sigma}$, i.e. at $t > 0$, equation (7) becomes:

$$TVO_t = \mathbb{E}_t \left[\overline{\sigma} \sqrt{T/(\epsilon_t + \langle X \rangle_T - \langle X \rangle_t)} C^{BS}(S_t, K, \langle X \rangle_t - \langle X \rangle_T) \right] \quad (17)$$

where $\epsilon = \langle X \rangle_t$.

The valuation formula loses its symmetry, because we shall this time multiplicate equation (8) by $\overline{\sigma}\sqrt{T}/\sqrt{x+\epsilon}$, giving rise in the Taylor development to terms of the form:

$$\frac{N(-\sqrt{x}/2)}{\sqrt{\epsilon + x}}, \frac{x^{-(j+1/2)}}{\sqrt{\epsilon + x}} \quad (18)$$

However, it is possible to use a second series expansion and develop the functions $N(-\sqrt{x}/2)$ and $x^{-(j+1/2)}$ in a neighbourhood of $x + \epsilon$ to restore such symmetry. This will produce a formula similar to (13), but accounting for a further summation and the presence of ϵ in the coefficients.

Robust pricing

In the influencial paper [2], Carr and Lee have proved that volatility derivatives can be statically replicated by claims on the underlying stock. This is thanks to the relation, which holds under the independence between the stock and volatility :

$$E_t[e^{\lambda(\langle X\rangle_T - \langle X\rangle_t)}] = E_t\left[e^{(X_T - X_t)p(\lambda)}\right] = E_t\left(\frac{S_T}{S_t}\right)^{p(\lambda)}, \qquad (19)$$

for some function p. In turn, this allows for model-independent (robust) pricing, that is, prices can be uniquely derived by liquid market data by means of the Breeden-Litzenberger formula:

$$f(S) = f(\eta) + f^{'}(k)[S-\eta] + \int_{\eta}^{\infty} f^{''}(x)(S-x)^{+}dx + \int_{0}^{\eta} f^{''}(x)(x-S)^{+}dx.$$

where f will be given through use of equation (19).
As it happens their results apply here because equation (7) involves the pricing of a pure derivative product .

The target volatility option: Laplace transform method

The TVO can also be priced by using the well-established methodology of the Laplace/Fourier transform inversion pionereed by Heston [6], Lewis [7] and Carr and Madan [1].

We write the TVO put valuation formula in terms of the log-strike k:

$$\bar{P}_t(S_t, e^k, \langle X \rangle_t) = \mathbb{E}_t \left[\frac{\bar{\sigma}\sqrt{T}}{\sqrt{\langle X \rangle_T}} (e^k - S_T)^+ \right] \equiv P(k). \tag{20}$$

By Laplace-transforming the put option price in the log-strike we get, for $\mathrm{Re}(\alpha) > 1$:

$$\begin{aligned} \hat{P}_t(\alpha) &\equiv \int_0^\infty e^{-\alpha k} P_t(k) dk \\ &= \bar{\sigma}\sqrt{T} S_t^{1-\alpha} \mathbb{E}_t \left[\frac{1}{\sqrt{\epsilon_t + \langle X \rangle_T - \langle X \rangle_t}} \frac{e^{(1-\alpha)(X_T - X_t)}}{\alpha(\alpha - 1)} \right]. \end{aligned} \tag{21}$$

The target volatility option: Laplace transform method

Again formula (12) comes to rescue us:

$$\frac{1}{\sqrt{\epsilon + x}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2(\epsilon+x)} dz. \tag{22}$$

Hence:

$$\begin{aligned} \hat{P}_t(\alpha) &= 2\bar{\sigma}\sqrt{\frac{T}{\pi}} S_t^{1-\alpha} \int_0^\infty e^{-z^2\langle X\rangle_t} E_t\left[\frac{e^{-z^2(\langle X\rangle_T - \langle X\rangle_t) + (1-\alpha)(X_T - X_t)}}{\alpha(\alpha-1)}\right] dz \\ &= 2\bar{\sigma}\sqrt{\frac{T}{\pi}} S_t^{1-\alpha} \int_0^\infty \frac{e^{-z^2\langle X\rangle_t} E_t\left[e^{\lambda_{z,\alpha}(\langle X\rangle_T - \langle X\rangle_t)}\right]}{\alpha(\alpha-1)} dz, \end{aligned} \tag{23}$$

In the second equality we have used equation (19). For most stochastic volatility models the Laplace transform of $\langle X\rangle_t$ is known, and decays fast enough to yield convergence of the integral.

The target volatility option: Laplace transform method

By inverting the transform we finally have:

$$P_t(k) = \frac{4e^{ak}\bar{\sigma}\sqrt{T}}{\pi^{3/2}} \int_0^{\infty}\int_0^{\infty} e^{-z^2\langle X\rangle_t}\mathrm{Re}\left(\frac{S_t^{1-a-iu}E_t\left[e^{\lambda_{z,a+iu}(\langle X\rangle_T-\langle X\rangle_t)}\right]}{(a+iu)(a+iu-1)}\right)cos(uk)dzdu. \quad (24)$$

We have applied a particular trigonometric representation of the Lapalce inversion which is use Abate-Whitt quadrature method.

The TVO call price is obtained by put-call parity with the target volatility forward:

$$TVOF_0 = \bar{\sigma}\sqrt{T/\langle X\rangle_T}(S_T - K) \quad (25)$$

whose valuation is straightforward after conditioning the value process to σ_T.

The target volatility strategy

A target volatility strategy is a dynamic portfolio allocation in an equity S_t and a bond B_t aimed at maintaining a constant portfolio volatility $\overline{\sigma}$ throughout the whole holding period $[0, T]$.

This is done by means of trades in the underlying occurring at given dates $t_1, \ldots, t_n$, of fixed lag δ . The volume of the trades at each period depends on the volatility realized in the previous period.

Let the equation for the bond and the stock be given by:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t \tag{26}$$

$$dB_t = rB_t dt \tag{27}$$

for some stochastic volatility process σ_t.

To construct the portfolio let Π_0 our intial endowment. Write Π_t in terms of number of portfolio holdings as:

$$\Pi_t^\delta = \Theta_t^\delta S_t + \Theta_t^r B_t = w_t^r \Pi_t^\delta + w_t^\delta \Pi_t^\delta$$

where, $w_t^r = \Theta_t^r B_t / \Pi\delta_t$, $w_t^\delta = \Theta_t^\delta S_t / \Pi_t^\delta$. Observe that $w_t^r + w_t^\delta = 1$.

Calculating the stochastic differential and assuming the self-financing property we have:

$$d\Pi_t^\delta = \Theta_t^\delta dS_t + \Theta_t^r dB_t = w_\delta^r \Pi_t^\delta dB_t / B_t + w_t^\delta \Pi_t dS_t / S_t = \quad (28)$$

$$r\Pi_t^\delta dt + w_t^\delta (\mu - r)\Pi_t^\delta dt + w_t^\delta \sigma_t \Pi_t^\delta dW_t \quad (29)$$

The target volatility strategy

Now, we want to set the time-varying (with δ and t) exposure w_t^δ to the stock component, in such a way that the log-returns of Π_t^δ show approximately constant volatility. Further, we want w_t^δ to depend only on observable market data.

This can be achieved by setting:

$$w_t^\delta = \overline{\sigma} \left(\frac{\delta}{\sigma_0^2 \delta \mathbb{1}_{\{0 \leq t < t_1\}} + \sum_{i \geq 1} I_{t_{i-1}}^{t_i} \mathbb{1}_{\{t_i \leq t < t_{i+1}\}}} \right)^{1/2} \tag{30}$$

with:

$$I_s^t = \int_s^t \sigma_u^2 du \tag{31}$$

Let $Y_t^\delta = \log \Pi_t^\delta$. We have the following proposition:

The target volatility strategy

Proposition

$Y_t^\delta \to Y_0^t$ in $L^2([0,T]\times\Omega)$ for $\delta \to 0$, where

$$Y_t^0 = (r - \overline{\sigma}^2/2)t + \int_0^t \frac{\overline{\sigma}(\mu - r)}{\sigma_u} du + \overline{\sigma} W_t \tag{32}$$

is the asymptotic portfolio of Π_t^δ.

Assume now that we want to write a derivative F on Y_t^δ. It is a consequence of the Proposition above that:

Corollary

Let F be a sufficiently regular European claim written on Y_t^δ. As $\delta \to 0$ the price of $F(Y_t^\delta)$ converges to $F^{BS}(\overline{\sigma})$.

In particular a TVS extends the property of the TVO to any moneyness.

Open problem 1 : the corollary above ensures that for fixed δ the price of a derivative contract on a TVS can be bound as:

$$\mathbb{E}[F(Y_t^{\delta})] \leq F^{BS} + \epsilon(\delta, parameters) \tag{33}$$

The error ϵ, which depends on δ and on the parameters of σ_t, represents from a practical persepctive the margin an operator must charge to an investor over the Black Scholes to underwrite the derivative F.

Is it possible to estimate, bound, or otherwise assess ϵ?

The target volatility strategy: distribution and moments properties

The expected return of the asymptotic portfolio is easy to compute, provided we know the first inverse moment of σ_t. Indeed

$$\mathbb{E}[Y_t^0] = \overline{\sigma}(\mu - r)\int_0^t \mathbb{E}[\frac{1}{\sigma_u}]du.$$

However, analytical formulae for the moments of $\int_0^t du/\sigma_u$ other than the first have, to our knowledge, not been derived, even for tractable stochastic volatility model (Dufresne, [4] has tried, but mentions insuccess).

Open problem 2: Can be such formulae found? If not, empirical testing supports the following evidence:

1. If the correlation between S_t and σ_t is negative (resp. positive) then the variance of Y_t^0 is higher (resp. lower) than that of the underlying risk-neutral Black-Scholes asset of standard deviation $\overline{\sigma}$ and return rate r;

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2. The skewness of Y_t^0 is positive; the higher the volatility of the volatility σ_t, the higher the skewness.
3. Numerically Y_t^0 shows excess kurtosis; however if $\mathrm{Corr}(S_t, \sigma_t) > 0$ then graphically the the tails of the distribution density seem to be at best Gaussian (which is counter-intuitive).

How to prove some of the above rigorously?

Heston model simulation, parameters from [8]

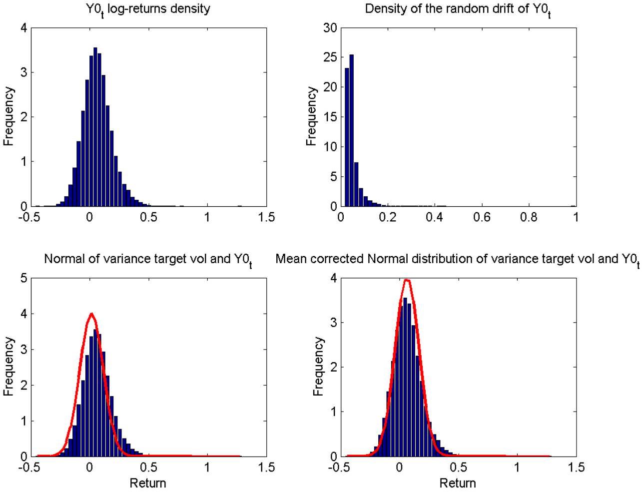

Figure : $\rho = -0.92$, vol of vol$= 0.1968$

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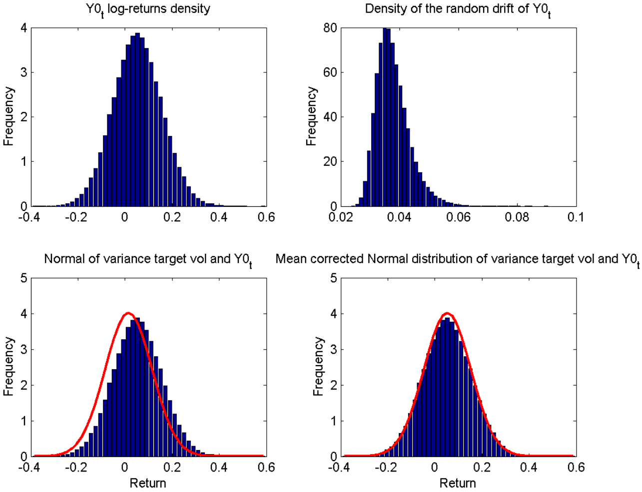

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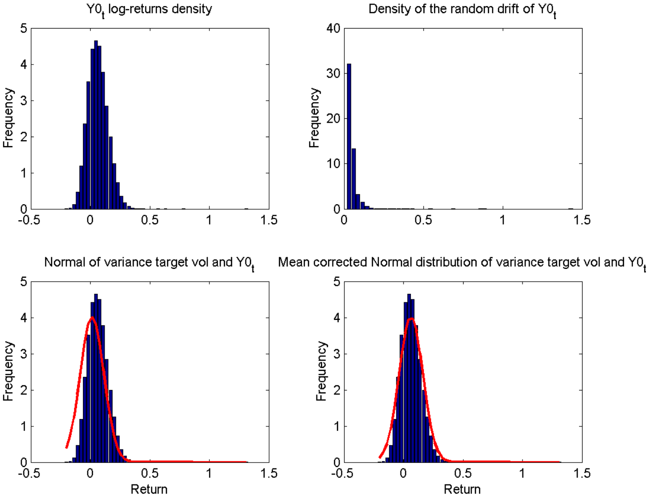

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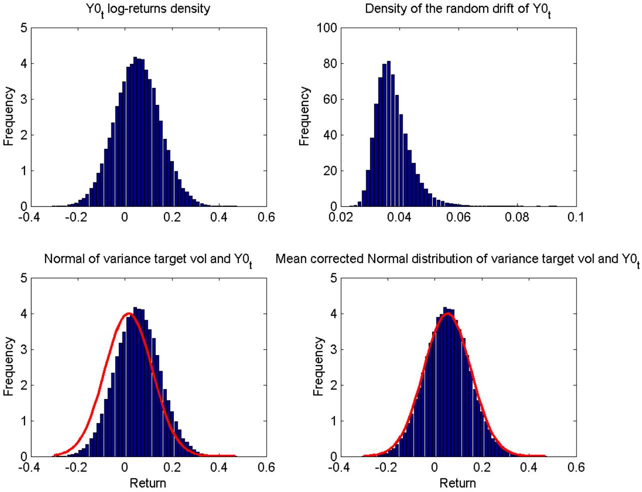

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Joint asset/volatility derivatives: the general problem

More generally to the TVO case we might want to consider general European derivatives that pay on acombination of S_t and $\langle X \rangle_t$.

These claims can be represented as a function $F(x, y)$ of two variables. Assume that the joint evolution of S_t, $v_t = \sigma_t^2$ is given by the following Ito diffusion:

$$\begin{cases} dS_t = (r-d)S_t dt + \sqrt{v_t} S_t dW_t^1 \\ dv_t = \alpha(t, v_t)dt + \beta(t, v_t)dW_t^2. \end{cases} \tag{34}$$

This can be augmented as to include $I_t = \langle X \rangle_t$ by considering the ODE:

$$dI_t = v_t dt. \tag{35}$$

The correlation between W_t^1, W_t^2 is arbitrary. Hence under this framework we manage to extend the TVO pricing as to include correlation.

Joint asset/volatility derivatives: the general problem

The Feynman-Kac theorem tells us that under appropriate assumptions the discounted risk-neutral expectation of $F(S_T, I_T)$ under the Markovian family of probability distibution generated by (34)-(35) is the solution of some parabolic PDE having terminal condition $F(S_T, I_T)$.

More precisely, and specifically:

$$V_t = \mathbb{E}_t[e^{-r(T-t)} F(S_T, I_T)] \tag{36}$$

is the solution to the degenerate PDE

$$\frac{\partial V}{\partial t} + (r-d)S\frac{\partial V}{\partial S} + \alpha\frac{\partial V}{\partial v} + v\frac{\partial V}{\partial I} + \frac{vS^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{\beta^2}{2}\frac{\partial^2 V}{\partial v^2} + \tag{37}$$

$$\rho\beta\sqrt{v}S\frac{\partial V}{\partial S \partial v} - rV = 0, \tag{38}$$

having terminal condition $F(S_T, I_T)$.

Joint asset/volatility derivatives: the general problem

Assume that the fundamental solution $\gamma(z, v, t)$ of the PDE associated to the diffusion (34) alone is known and let $\Gamma(z, w)$ be instead the fundamental solution of (37).

Notably, for most volatility modles we have that there exists $G : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ such that:

$$\Gamma(z, w, v, t) = G(g(z, w), v, t) \tag{39}$$

$$\gamma(z, v, t) = G(g(z, 0), v, t) \tag{40}$$

Finally, the following pricing equation holds true:

Proposition

Under certain constraints on (34) the solution of (37) is given by:

$$V(S, I, v, t) = \frac{e^{-r(T-t)}}{4\pi^2} \times \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} S^{-iz} e^{-iz(r-d)(T-t)} e^{-iwI} \hat{\Gamma}(z, w, v, t) \hat{F}(z, w) dz dw. \tag{41}$$

Valuation of the pricing integral

The numerical evaluation of integral (41) poses some difficulties, having a two-dimensional complex integrand. These can be both due to the oscillation and convergence rate along the choosen line.

There is a typical strategy to overcome these problems, allowed by the the Cauchy theorem: the saddle point method. We deform the integration countour to γ for which there exists a point $p \in \gamma$ such that:

1. The imaginary part of the integrand has a stationary point in p: this reduces the oscillation;

Open problem 3: Any way of determining γ for, say, the TVO payoff and the Heston model?

Valuation of the pricing integral

The numerical evaluation of integral (41) poses some difficulties, having a two-dimensional complex integrand. These can be both due to the oscillation and convergence rate along the choosen line.

There is a typical strategy to overcome these problems, allowed by the the Cauchy theorem: the saddle point method. We deform the integration countour to γ for which there exists a point $p \in \gamma$ such that:

1. The imaginary part of the integrand has a stationary point in p: this reduces the oscillation;
2. The real part of the integrand has a local maximum in p; this means that the mass is mostly around p and there is less computational effort to get the prescribed accuracy.

Open problem 3: Any way of determining γ for, say, the TVO payoff and the Heston model?

Joint asset/volatility derivatives: some fantasy payoffs

Following are some more payoffs of the type $F(S_T, I_T)$, which do not exist but could make sense as real-life investments:

1. Double digital call:

$$F(S_T, I_T) = \mathbb{1}_{\{S_T \geq K_1,\, I_T/T \geq K_2\}}; \tag{42}$$

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3. Volatility-struck call option:

$$F(S_T, I_T) = \left(S_T - N\sqrt{\frac{I_T}{T}} \right)^+. \tag{44}$$

Thank you

All criticism, comments and suggestions welcome:

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P. Carr, D. Madan, and R. H. Smith.
Option valuation using the fast fourier transform.
Journal of Computational Finance, 2:61–73, (1999).

P. Carr and R. Lee.
Robust replication of volatility derivatives.
Mathematics in finance working paper series, New York University, Courant Institute of Mathematical Sciences, (2008)3.

G. Di Graziano and L. Torricelli.
Target volatility option pricing,
International Journal of Theoretical and Applied Finance **15**(1) (2012).

Daniel Dufresne.
The Integrated Square-Root Process.
University of Montreal Research Paper 90 (2001).

A. Estrella.
Taylor, Black and Scholes: series approximations and risk management pitfalls.
Research Paper 9501, Federal Reserve Bank of New York, (1995).

S. L. Heston,
A closed-form solution for options with stochastic volatility with applications to bond and currency options,
The Review of Financial Studies **6**(2) (1993) 327–343.

A. Lewis.
Option Valuation Under Stochastic Volatility
Finance Press (2000).

S. Stoyanov.
Structured Equity Investments for Long-Term Asian Investors.
EDHEC-Risk Institute Publications, (2011).

L. Torricelli.
Pricing joint claims on an asset and its realized variance in stochastic volatility models.
International Journal of Theoretical and Applied Finance, **16**(1) (2013) . Preprint retrievable at *ArXiv* id: 1206.2112v1 [q-fin.PR].