Valuation of asset and volatility-dependent derivatives using decoupled time-changed Lévy processes

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Workplan

1. Aim of the work

In a market where investors have risk-neutral preferences:

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In a market where investors have risk-neutral preferences:

1. We introduce a novel asset model representation based on time changes
2. We pose and solve the problem of pricing European-style derivatives paying jointly on an asset S_t and its realized volatility $RV_t = \langle \log S_t \rangle$

2. Time changes and decoupled time changes

Given a Lévy process X_t and a.s. finite cadlag process T_t, the time change of X_t by T_t is the process X_{T_t}. It has been long conjectured (Barndoff-Nielsen, Carr, Madan, Yor, Geman and MANY others) yield to very realistic asset models.

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- The most popular models are the Lévy subordinated models where X_t is a Brownian motion and T_t is an increasing Lévy process independent of X_t.
- In their 2004 paper [**?**], Carr and Wu showed that completely general processes X_{T_t} generate asset price models compatible with the risk-neutral theory.

If X_t is a Lévy process, it can be uniquely decomposed as the (independent) sum

$$X_t^c + X_t^d \tag{1}$$

where X_t^c and X_t^d are respectively the continuous and pure jump parts of X_t. Therefore a time change X_{T_t} admits the representation:

$$X_{T_t} = X_{T_t}^c + X_{T_t}^d \tag{2}$$

But now what happens if we let one of the two components to be time changed according to a time scaling U_t different (and not necessarily independent) from T_t?

Answer: by imposing technical assumption, we get a new process

$$X_{T,U} := X^c_{T_t} + X^d_{U_t} \tag{3}$$

that is still suitable for asset price modeling.

I have called $X_{T,U}$ a **decoupled time changed (DTC) Lévy process** .

The a technical assumption required is continuity with respect to the time change: that is, X^i_t must be constant on all the sets $[X^i_{T^i_t-}, X^i_{T^i_t}]$.
A sufficient condition to enforce this is to take as T^i_t the pathwise integral:

$$T_t = \int_0^t v^{i-}_s ds,$$

for some cadlag process v_t called the instantaneous activity rate

- Counterxample: A Brownian subordinated model is not a decoulped time change.

We have the following result:

Proposition

Let X_t^1 be an n-dimensional Brownian motion with drift and X_t^2 a pure jump Lévy process in $\mathbb{R}^n$. Let T_t^1 and T_t^2 be two time changes such that X_t^1 and X_t^2 are respectively T_t^1 and T_t^2-continuous. Set $X_t = X_t^1 + X_t^2$ and $T_t = (T_t^1, T_t^2)$; define $X_{T_t} := X_{T_t^1}^1 + X_{T_t^2}^2$ and Θ be the domain of definition of $\mathbb{E}[\exp(i\theta^T X_t^2)]$. The process:

$$M_t(\theta, X_t, T_t) = \exp\left(i\theta^T X_{T_t} - (T_t^1 \psi_{X^1}(\theta) + T_t^2 \psi_{X^2}(\theta)\right) \tag{4}$$

is a local martingale, and it is a martingale is a local martingale, and it is a martingale if and only if $\theta \in \Theta_0$, where:

$$\Theta_0 = \{\theta \in \Theta \text{ such that } \mathbb{E}[M_t(\theta, X_t, T_t)] = 1, \forall t \geq 0\}. \tag{5}$$

In particular for $\theta_0 \in \Theta_0$ and for a given spot price value S_0 we can define an asset directly by its risk-neutral dynamics by:

$$S_t := S_0 \exp(rt + i\theta_0 X_{T,U} - c). \quad (6)$$

Indeed, by construction, the discounted asset price and discounted prices of derivatives written on S_t, are martingales.

DTC Lévy processes encompass in a unitary framework many previous available asset price models coming from apparently distant asset classes.

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- For the Bates model [**?**] we have X_t^c is a Wiener process, X_t^d a compound Poisson process with normally distributed jumps, $T_t = \int_0^t \sigma_s^2 ds$ and $U_t = t$ for a stochastic volatility process σ_t;

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- The stochastic volatility/stochastic jump rates models (e.g. Fang, [**?**]) X_t^c is a Wiener process, X_t^d a compound Poisson process, $T_t = \int_0^t \sigma_s^2 ds$ and $U_t = \int_0^t \sigma_s^2 ds$ for a stochastic volatility process σ_t and stochastic jump rate λ_t;

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  The last two are examples of DTC process that are not time-changed in the ordinary sense.

4. Characteristic functions and the inverse-Fourier pricing method

Lewis ([?], [?]) and many others (FFT, ecc.), have demonstarated that the characteristic function of the log price process is a powerful ingredient to obtain semi-analytical pricing formulae.

Let Y_t a stochastic process to be specified, and $\Phi(z) = \mathbb{E}[exp(i\theta Y_t)]$ its characteristi function.

If $\log S_t = rt + Y_t$ then the price of a derivative $F(\log S_t)$ is:

$$\mathbb{E}[e^{-rt}F(\log S_t)] = \frac{e^{-rt}}{2\pi}\int_{ik-\infty}^{ik+\infty} S_0^{-iz}e^{-rt(iz)}\Phi(-z)\hat{F}(z)dz. \qquad (7)$$

where $\hat{\cdot}$ indicates the Fourier transform, and the integral is taken in a line where both $\hat{F}$ and Φ are analytical.

For instance Φ can be explicitly derived when Y_t is a Lévy processes, or if its density is the fundamental solution of a parabolic PDE.

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In the present work, we are interested in understanding if (**??**) can be generalized when:

1. $X_t = X_{T,U}$ is a decoupled time-changed Lévy process;
2. The derivative we price does not depend only on the asset value S_t but also from its realized volatility RV_t. That is we consider claims of the form

$$F(S_t, \langle \log S_t \rangle) \tag{8}$$

for some payoff function F of two real variables.

Indeed it turns out that under these circumstances the pricing formula **??** can be extended as follows:

Theorem

Let $\log S_t$*, with* S_t *given by (**??**). Let* $F(x,y) \in L^1(\log S_t, \langle \log S_t \rangle)$ *and* $\Phi(z,w) = \mathbb{E}[\exp(izX_{T,U} + iw\langle X_{T,U}\rangle)]$ *be the joint characteristic function of* $X_{T,U}$ *and* $\langle X_{T,U}\rangle$*. The value of the contingent claim* F *maturing at time* t *is:*

$$\mathbb{E}[e^{-rt)} F(Y_t, \langle Y \rangle_t)] = \frac{e^{-rt)}}{4\pi^2} \cdot \int_{ik_1-\infty}^{ik_1+\infty} \int_{ik_2-\infty}^{ik_2+\infty} S_0^{-iz} e^{-irtz} \Phi_(- z, -w) \hat{F}(z,w) dz dw. \qquad (9)$$

Therefore, to completeley solve the pricing problem presented, everything boils down to finding Φ. If X_t has Lé vy characteristics (μ, σ, ν) a lengthy but rather straightforward computation shows that

$$\Phi(z, w) = \mathcal{L}^{\mathbb{Q}}_{T_t, U_t}(\theta_0 \mu(z - iz) - \theta_0^2 \sigma^2 (z^2 + iz - 2iw)/2, iz\psi_X^d(\theta_0) \quad (10)$$
$$- \psi_D(iz\theta_0, iw\theta_0)) \quad (11)$$

Here $D_t = (X_t^d, i\theta_0 \langle X \rangle_t^d)$ and $\mathcal{L}^{\mathbb{Q}}_{T_t, U_t}$ is the Laplace transform taken with resepct of a ¶- equiavalent measure $\mathbb{Q}(z, w)$ which is called the leverage neutral measure (Carr and Wu 2004).
Pricing through formula **??** is then possible by only knowing the joint distribution of T_t, U_t under $\mathbb{Q}(z, w)$, which is usually recovered from that of $\mathbb{Q}$ through Girsanov's Theorem.

Model-specific formulae for Φ

Here are some explicit expressions for Φ

- The Black-Scholes model:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{12}$$

$$\Phi(z, w) = \exp(-t\sigma^2(z^2 + iz - 2iw)/2). \tag{13}$$

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- Heston model:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^1; \tag{14}$$

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t} dW_t^2. \tag{15}$$

with $\langle W_t^1, W_t^2 \rangle = \rho t$.

$$\Phi(z, w) = \mathcal{L}^z_{T_t}(z^2/2 + iz/2 - iw) \tag{16}$$

where $T_t = \int_0^t v_s^z ds$, $dv_t^z = \alpha^z(\theta^z - v_t)dt + \eta\sqrt{v_t} d\hat{W}_t$ and $\alpha^z = \alpha - i\rho z\eta$, $\theta^z = \alpha\theta/\alpha^z$ (compare [?], 2012)

- Jump diffusion models:

$$dS_t = rS_{t^-}dt + \sigma_t S_{t^-}dW_t + S_{t^-}JdN_t - \kappa\lambda S_{t^-}dt$$

$J \sim N(\mu, \delta^2)$ is Merton's model, $J \sim DblExp(\alpha, \beta)$ is Kou's. For both model we have:

$$\Phi(z, w) = \exp(t(\sigma^2(-z^2/2 - iz/2 + 2iw)/2+ \quad (17)$$
$$\lambda(\phi_{J,J^2}(z, w) - iz\phi_J(-i) + iz - 1)). \quad (18)$$

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- CGMY (Carr, Madan, Geman, Yor-s) Lévy model is the infinite activity jump process having Lévy density:

$$\frac{d\nu(x)}{dx} = \frac{c_- e^{-\beta_-|x|}}{|x|^{1+\alpha_-}}\mathbb{1}_{\{x<0\}} + \frac{c_+ e^{-\beta_+ x}}{x^{1+\alpha_+}}\mathbb{1}_{\{x\geq 0\}}. \quad (19)$$

for constants $c_\pm, \beta_\pm, \alpha_\pm$.

it is:

$$\Phi(z,w) = \exp(t(\psi_D(z,w) - iz\psi_X^d(-i))). \tag{20}$$

where:

$$\psi_X^d(\theta) = \gamma_1 + \Gamma(-\alpha_+)\beta_+^{\alpha_+} c_+ \left(\left(1 - \frac{i\theta}{\beta_+}\right)^{\alpha_+} - 1 + \frac{i\theta\alpha_+}{\beta_+} \right) + \Gamma(-\alpha_-)\beta_-^{\alpha_-} c_- \left(\left(1 + \frac{i\theta}{\beta_-}\right)^{\alpha_-} - 1 - \frac{i\theta\alpha_-}{\beta_-} \right),$$

$$\psi_D^+(z,w) = iz\gamma_1 + iw\gamma_2 + \int_0^{+\infty} (e^{izx+iwx^2} - 1 - (izx + iwx^2)) \frac{c_+ e^{-\beta_+ x}}{x^{1+\alpha_+}} dx =$$

$$ic_+\beta_+^{\alpha_+} \left(-w\frac{\Gamma(2-\alpha_+)}{2i\beta_+^2} - z\frac{\Gamma(1-\alpha_+)}{2i\beta_+} + i\Gamma(-\alpha_+) \right) - c_+(\beta_+ - iz)^{\alpha_+} \left(\frac{i(\beta_+}{} \right.$$

$$\left(\sqrt{\frac{i(\beta_+ - iz)}{w}} \Gamma\left(\frac{1}{2} - \frac{\alpha_+}{2}\right) {}_1F_1\left[\frac{1-\alpha_+}{2}, \frac{3}{2}, \frac{i(\beta_+ - iz)^2}{4w}\right] - \right.$$

$$\left. \Gamma\left(-\frac{\alpha_+}{2}\right) {}_1F_1\left[-\frac{\alpha_+}{2}, \frac{1}{2}, \frac{i(\beta_+ - iz)^2}{4w}\right] \right). \tag{21}$$

- Bates model:

$$dS_t = rS_{t^-} + \sqrt{v_t}S_{t^-}dW_t^1 + S_{t^-}JdN_t - \kappa\lambda S_{t^-}dt; \quad (22)$$

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2. \quad (23)$$

$$\Phi(z,w) = \mathcal{L}^z_{T_t}(z^2/2 + iz/2 - iw))\exp(t\lambda(\phi_{J,J^2}(z,w) - iz\kappa - 1). \quad (24)$$

Compare with (**??**) and (**??**). T_t is the usual integrated variance. That is, the Bates model can be described as a DTC jump diffusion where only the continuous part has been time changed.

- Fang model:

$$dS_t = rS_{t^-}dt + \sqrt{v_t}S_{t^-}dW_t^1 + S_{t^-}JdN_t - \kappa\lambda_t S_{t^-}dt;$$

$$dv_t = \alpha(\theta - v_t)dt + \eta\sqrt{v_t}dW_t^2;$$

$$d\lambda_t = \alpha_\lambda(\theta_\lambda - \lambda_t)dt + \eta_\lambda\sqrt{\lambda_t}dW_t^3.$$

$$\Phi(z,w) = \mathcal{L}^z_{T_t}(z^2/2 + iz/2 - iw)\mathcal{L}_{U_t}(iz\kappa - \phi_{J,J^2}(z,w) + 1). \quad (25)$$

Fang model is again DTC jump diffusion where both the continuous and discontinuous parts have been time changed.

Some numbers

The striking fact about pricing equation (**??**) is that, by voiding the relevant diffusion parameters a single software implementation is able to capture a vast range of different models and asset classes.

We have implemented it for the models given above and for three different instances of asset classes: an vanilla asset derivative (Call Option), a pure volatility derivative (volatility call option), and a joint asset/volatility derivative (the Target Volatility Option; see [**?**], [**?**]).
The Fourier Transforms of the relative payoffs to be used are:

- Call option: $F(z) = (e^z - K)^+$

$$\hat{F}(z) = \frac{K^{1+iz}}{(iz - z^2)};$$

- Volatility call option $F(w) = (\sqrt{w} - Q)^{+}$

$$\hat{F}(w) = \frac{\sqrt{\pi}\ \mathrm{Erfc}(Q\sqrt{-iw})}{2(-iw)^{3/2}}; \tag{26}$$

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$$\hat{F}(w) = \frac{\sqrt{\pi}\ \mathrm{Erfc}(Q\sqrt{-iw})}{2(-iw)^{3/2}}; \tag{26}$$

- Target volatility call option $F(z, w) = \bar{\sigma}\sqrt{\frac{t}{w}}(e^z - H)^+$:

$$\hat{F}(z, w) = \bar{\sigma}(1 + i)\sqrt{\frac{\pi t}{2w}}\frac{H^{1+iz}}{(iz - z^2)}. \tag{27}$$

Here is a tests. A MATHEMATICA implementation pricing equation has been compared to an Euler scheme for the Monte Carlo simulation of the option values.

Table : $S_0 = 100,\ K = H = 120,\ Q = 0.1,\ t = 3.5,\ r = 0.039,\ \overline{\sigma} = 0.1,$ $TV_{t_0} = 0.018.$

Model	Vanilla call		Volatility call		
	AV	MC	AV	MC	A\
Black-Scholes	8.4801	8.4784(0.02%)	0.1672	0.1695(1.36%)	5.76
Heston	10.3063	10.3023(0.04%)	0.2167	0.2172(0.23%)	6.38
Merton	11.5845	11.5713(0.11%)	0.2357	0.2356(0.04%)	7.45
Bates	9.8607	9.8371(0.24%)	0.2002	0.2001(0.05%)	6.81
Fang	8.8630	8.8737(0.12%)	0.1827	0.1828(0.05%)	7.41

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